Ten Years of Using Symbolic TI Calculators at ETS

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Starting in September 1999, new students at ETS were required to own the TI-92 Plus or TI-89 symbolic calculator and since September 2002 the Voyage 200. Looking back at these ten years of working with a computer algebra system on every student's desk, one could ask whether the introduction of this hand-held technology has really forced teachers to reassess their goals in teaching mathematics. For some teachers – in fact, fewer than expected − the answer is "yes". But what has really changed? Although some exam questions are different because students almost always have access to their calculator during tests, the curriculum is essentially the same. Many professors still continue to ignore the power of computer algebra; they consider it merely a substitute for pencil and paper techniques or a way to illustrate concepts. This paper will give our perspective on this ten-year experiment based in part on many discussions with colleagues as well as positive feedback from students, who have often thanked us for showing them how to use their CAS calculators efficiently. We will give examples of what daily use of computer algebra in the classroom should produce: a better appropriation of (many) mathematical concepts. Computer algebra systems are here to stay. Not using them won't make them disappear.

1   INTRODUCTION

After ten years of symbolic calculator use at ETS, we have found that the mathematics curriculum remains relatively unchanged. This is easy to see if we take a look at some mathematics courses in our ETS engineering school. Because the students have access to a CAS, they can deal with more application problems, and this is probably one of the major changes over the past 10 years. With a CAS, many teachers − but not a majority − have started to give their students more challenging problems. At the same time, however, not more − even less –fundamental and basic mathematics was presented, as if it was impossible to teach mathematics when using a CAS. Here are some examples of what we mean by "fundamental and basic mathematics". Complete examples will be given in the next section.

a) In Mat 144 (Pre-calculus), teachers review some properties of real numbers, rules of exponents or logarithms, various formulas concerning square roots, and trigonometric identities. Many of them do not take advantage of the CAS to extend these properties to complex numbers. This would be understandable if ETS were a high school; however it is a university engineering school. Some colleagues don't accept the fact that, even for basic mathematics, the CAS calculator can be used to learn, teach and do mathematics.

b) In Mat 145 (Single-variable Calculus), fewer subjects are currently studied despite the fact that every student is equipped with a Voyage 200! Some integration techniques have even been removed from the curriculum. There is no reason for this. In fact, the availability of the CAS provides the opportunity to compute more integrals and explain to students how the system performs these calculations; of course, this may be different than how they might do it by hand. In this course we only have 13 weeks to scroll through all the classic single-variable calculus. We therefore have to be efficient and put emphasis on tools needed to solve applied problems.

c) In Mat 165 (Multiple Variable Calculus/Linear Algebra), the Voyage 200 calculator can't compete with a major CAS running on computers when plotting 3D surfaces/curves. However, the solving facilities of this CAS calculator should be used to facilitate optimization problem solving with or without constraints. The 2D parametric plotting window is rarely used by teachers. Even if the 3D plotting facilities of the V200 are very limited, teachers can nevertheless show their students what a level curve is, explain the importance of 2D implicit plotting, and, more generally, introduce other 3D plotting aspects (explicit, implicit, parametric).

d) In Mat 265 (Differential Equations), some colleagues continue to teach many technical methods for solving different kinds of DEs but at the end of the course, many students have no idea how to produce a slope field or plot the curve using some numerical methods. Looking at the method of undetermined coefficients, many students continue to apply it by hand instead of automating it. Another important concept studied in Mat 265 is the Laplace transform: the fact that the V200 does not possess an inverse Laplace transform function can be seen as an advantage. Time can be spent on using the "expand" function and completing squares where necessary and/or the convolution theorem, thus doing part of the classical approach using the CAS for calculations.

2   CONCRETE SUGGESTIONS FOR EACH OF THE FOUR CHOSEN COURSES

Example 1 (from "Pre-calculus", Mat 144 at ETS)

A pre-calculus review course, within an era of computer algebra, could be much more interesting; for example, while reviewing many results and formulas, the instructor should tell the students that they can do their own study with the help of their CAS calculator. To illustrate this, consider in figure 1 the F2 "Algebra" menu of the V200:

Figure 1

Can we really teach mathematics with the V200 and never say a word about the "factor" command? In the "Catalog" of the V200, we see the syntax for the "factor" command (see figure 2). By default, it is simply "factor(expression)". Adding the variable will control the level of factorisation. In a pre-calculus course, we think that students should learn by using their V200 what the difference is in factoring $x^2 - 4$ and $x^2 - 5$ (see figure 3).
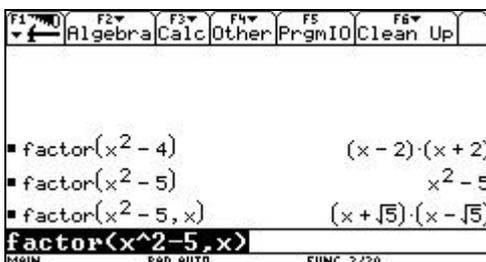

Figure 2

Now, consider the "propfrac" command. We can rapidly find the equation of the oblique asymptote to some rational function. Do students today still need to know how to do manually long division of polynomials? Consider this rational fraction: $\dfrac{3x^3 + x^2 + 3x + 3}{x^2 + 1}$. If the V200 does the division, one can focus on the interpretation of the result (see figure 4) and talk about limits and oblique asymptotes.
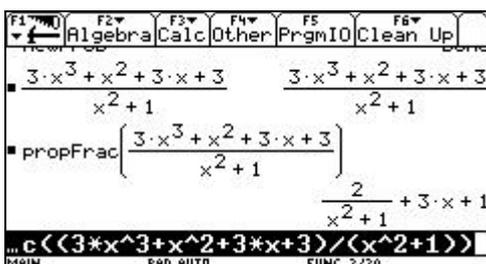

Figure 3

Figure 4

For the past 10 years, students have been using trigonometric identities and continue to do so. Let us take the example of "trig" commands, which can be used at different levels: addition formulas, double angle formulas and so on. In conjunction with Euler's formula and binomial expansion, one can easily show where some identities come from. This type of application is what we consider legitimate mathematics. (See figures 5 and 6.)
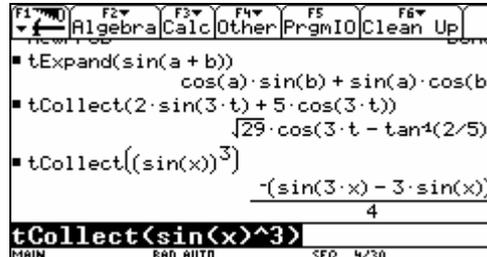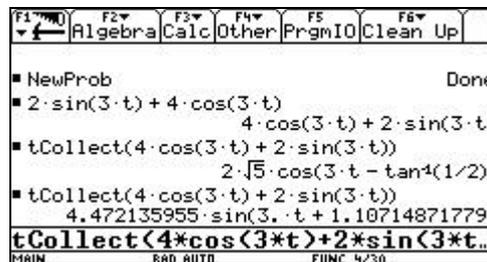

Figure 5

Figure 6

Looking at figure 6, we see two different results for the same command, one in "Exact" mode and the other in "Approx" mode. This transformation is important for students who will be studying harmonic motion. They need to find the amplitude and the phase angle of the movement. The two equivalent results obtained in figure 6 provide a good opportunity for teachers to do more mathematics.

If you ask students to find the cubic root of -8, they have no problem giving back the desired response, -2. Then when they see the result obtained by the V200, see figure 7 below, we have another example of technology giving us more mathematics to consider. This is a great opportunity to discuss complex numbers and to show students the "Complex format" options available in the mode menu of the calculator (see figure 8). By default, this option is set to "Real", which gives the expected answer. Of course, our calculator in the classroom is set to "Rectangular", which gives answers in the principal branch, and thus the complex number answer below.
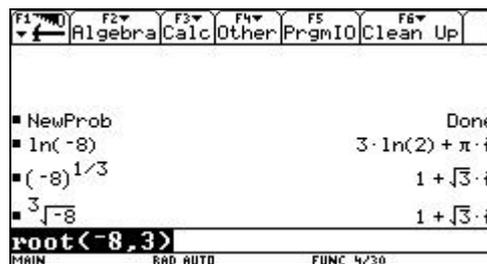

Figure 7

Figure 8

High schools students have learned the "rules of exponents" like $\left(a^x\right)^y = a^{xy}$. Because the order of the exponent x and y does not matter when a is positive, they are surprised when they face a negative value for a. Try $(-1)^{\frac{1}{3}}$ and then try $\left((-1)^4\right)^{\frac{1}{3}}$ and $\left((-1)^{\frac{1}{3}}\right)^4$ with the "Rectangular" option for "Complex Format". This is a great opportunity to discuss the complex branch and should also be part of the curriculum in a pre-calculus course where a mandatory CAS tool is required.

The "Solve" command on the V200 merits some attention. Looking at figure 9 below, one should realize that using this technology to solve equations requires doing quite a bit of mathematics to explain the results below (not the first one which is a typical pen and paper problem, but the others, which require explanations: note the warning at the bottom of the screen).
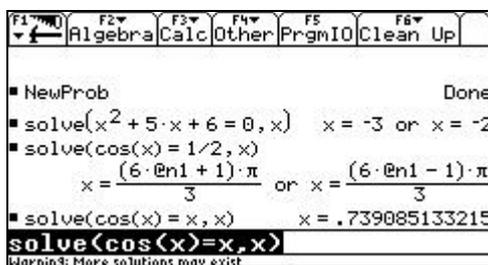

Figure 9

Example 2 (from "Calculus 1", Mat 145 at ETS).

Here, we will focus on two specific aspects. The first one concerns graphing modes. The V200 has 6 different graphing windows and, of course, the "Function" graphing mode is the most often used in calculus of one variable:

Figure 10

We believe that students attending an introductory calculus course in an engineering school should learn that the "Parametric" graphing mode can be useful to plot the graph of a function defined by $x = g(y)$. Very rarely, students are informed of how the "Sequence" mode can be used. This is curious because sequences and series are included in the Mat 145 curriculum. The "Function" 2D plot window of the V200 is used in Calculus 1 to visualize many concepts ─ min/max, inflection points, tangent line, area under the curve, intersection of 2 curves, etc. These are correct but would be also possible with a graphing calculator. Because the V200 is a CAS calculator, we encourage teachers to do the following exercise with their students. Using the "solve" command, find the 3 real solutions of the equation $2^x = x^{10} - 3$. The "Warning: More solutions may exist" message that appears on the bottom of the screen requires some explanation. The instructor should also explain the difference between "Exact" and "Auto" modes. Also, this will be a good opportunity to emphasize the fact that, eventually, the exponential will dominate the power ─ why are we teaching the l'Hôpital rule? An appropriate table of values will convince the students that a third solution exists, one that can be found rapidly using the "nsolve" command with a good starting point. Doing all of this is doing mathematics, using symbolic, graphic and numeric approaches.

The second point is integration techniques. Having access to a CAS should be an opportunity to continue to teach many techniques of integration ─ not fewer ─ with the idea that engineering students will very rarely have to integrate by hand. However, a student using a CAS calculator to compute $\int \frac{dx}{1+x^2}$ and $\int \frac{dx}{1+x^3}$ should know that these two integrals are not computed in the same manner. In other words, we would suggest the following approach regarding integration techniques.

a) To continue to pay attention to the change of variable methods. and to continue to present the integration by parts method but to explain to the students that this will be most often used to make proofs in some theorems or properties.

b) To continue to practice square completion for integrals of the form $\int \frac{dx}{ax^2 + bx + c}$ with $b^2 - 4ac < 0$.

c) Taking advantage of access to "texpand" and "tcollect" commands, to introduce integration of powers of sines and cosines (at least). If students are required to compute $\int \sin^4(x)\,dx$ with the V200, they will get an answer that looks different from the one they will obtain if they first replace the power of sin and then integrate. This is a good way to emphasize that two antiderivatives can differ by a constant.

d) Partial fraction expansion has been removed from the curriculum. Isn't it cumbersome to compute $\int \frac{4x-3}{x^2 + 5x + 6}\,dx$ by completing the square of the

denominator. It would be better to show students how to do this simple partial fraction expansion manually and then make use of the "expand" command. In this way, "heavy" rational functions would be given to the students for integration with the help of the "expand" command. This will reinforce using basic integration formulas, square completion, and change of variables. Revisiting an earlier example, our students can easily do the following integral $\int\left(3x+1+\dfrac{2}{x^2+1}\right)dx$, but for lack of time to see or review how to do long division, we do not ask them to do this one $\int\left(\dfrac{3x^3+x^2+3x+3}{x^2+1}\right)dx$. Considering that the "ProcFrac" command of the V200 would give them the simplified expression of the first integral, we lose the opportunity of teaching them how to deal with more complex integrals.

Example 3 (from "Calculus 2", Mat 165 at ETS).

Because the 3D plotting facilities of the V200 are very limited, many professors simply won't talk about the 3D window features of the V200. When students first meet a scalar field like the one defined by $f(x,y)=x^2+4y^2$, it would be a good idea to try to reproduce the graph they see from their book. This can be done quite easily and, in so doing, students will learn how to find an appropriate window to get the best picture. They will have to practice with the different parameters. When they will try to do the same on a computer, with a faster processor and even with colors, they will be more prepared: see figure 11 where we have chosen $z_1(x,y)=x^2+4y^2$:
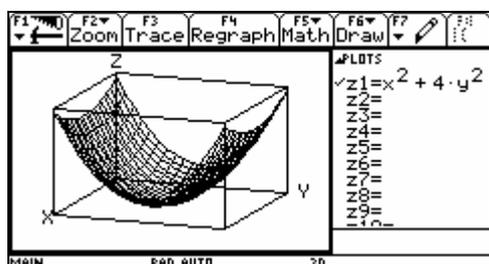

Figure 11

In the Mat 165 course, some colleagues think that the V200 can only be used for computing double and triple integrals, after students have found the integration limits. What about min/max problems? What about automating the search and analysis of local critical points? The V200 can solve, using Gröbner bases, polynomial systems in exact mode if possible; thus, given a 2-variable polynomial function or expression f, the TI won't forget any critical points. When it comes time to analyze it, few professors use the calculator in a "CAS environment". We don't understand why the critical points are not stored into a matrix using the "zeros" function instead of the "solve" function, nor do we understand why the "second partials test" is not done with the CAS calculator. It is easy to define the hessian matrix of f and then compute the discriminant of f and $\dfrac{\partial^2 f}{\partial x^2}$ at a particular critical point (a, b):

$$\text{zeros}\left(\left\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\},\{x,y\}\right)\ \text{STO}\ \text{ptcri}(f)$$

$$\det\left(\begin{bmatrix}\dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y}\\[2mm] \dfrac{\partial^2 f}{\partial x\partial y} & \dfrac{\partial^2 f}{\partial y^2}\end{bmatrix}\right)\ \text{STO}\ \text{hessien}(f)$$

$$\lim\left(\lim\left(\left[\text{hessien}(f),\partial^2 f/\partial x^2\right],x,a\right),y,b\right)\ \text{STO}\ \text{nature}(f,a,b)$$

Some professors argue that doing this won't allow the students to compute the partial derivative by hand, with the calculator doing everything. Instead students can be required to show that a particular critical point is a saddle point by producing an appropriate graph. Our point is, again, to use computer algebra as a tool for teaching and learning. In the classroom, we define some simple functions like the following ─ here, for simplicity, we are always satisfied with the letters x and y: of course, more general definitions can be made with more advanced students. When using computer algebra in teaching, instructors should never forget to start with simple, not complicated, definitions.

Now, give students a particular fourth degree polynomial (2 variables), say $x^4-xy^2+3y^3-5y$. The above functions will rapidly inform them that there are 2 critical points and one, namely the point $(0.49,-0.69)$, is a saddle point. The students can be asked to produce a graph where the saddle point can be seen clearly. On the V200, that will take time but the exercise is important in order to understand what 2D implicit plotting is.

Let's move to some linear algebra. There is no problem asking students to perform row operations by hand for small size matrices. Computer Algebra Systems can compute determinants, find the unique row-reduced-echelon form of a given (numeric) matrix, and find the inverse of a square non-singular matrix. Given this, time can now be spent on understanding the results obtained instead of doing by hand the operations to obtain these results. This can open the field for new kinds of problems, for example, when dealing with matrices having one (or more) parameter: the command "rref" can produce incorrect results, so this becomes a good opportunity to see that CAS, being programmed by humans, are like humans and can make mistakes or give strange results, see Beaudin/Picard [2]. Here is a good example, adapted from Jeffrey/Corless [3]: consider the linear system $\mathbf{AX}=\mathbf{B}$, where

$$\mathbf{A}=\begin{bmatrix}1 & -2 & 3\\ 1 & 4\cos(\alpha) & 3\\ -1 & 3 & \cos(\alpha)-3\end{bmatrix}\ \mathbf{X}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\ \mathbf{B}=\begin{bmatrix}\sin(\alpha)\\ 3\sin(\alpha)\\ \cos(\alpha)\end{bmatrix}$$

Here, α is a parameter, between 0 and 2π. Students will try to solve this problem using their V200 with the command "rref" to the augmented matrix $[\mathbf{A}\vdots\mathbf{B}]$. If they don't think to compute the determinant of $\mathbf{A}$ to see for which

values of α no unique solution exists, the "rref" command will give incomplete information.

Example 4 (from "Differential Equations", Mat 265 at ETS).

If there is a place where CAS has had an effect on the curriculum, it is in differential equations courses. Not only do CAS provide teachers the opportunity to empower students to deal with more realistic applications, but they can also help students get a better understanding of the meaning, theory, and techniques of differential equations. Here, we will focus on two specific items.

a) Why are we still asking our students to solve differential equations if they don't have any idea of the solution? In fact, where is that solution defined? Is the solution a unique "function"? If I can't solve the ODE by hand (or with my CAS), can I obtain a numerical solution? Although these questions should be part of exam questions in a CAS environment, this is often not the case, and many professors simply don't explore these avenues. For example, they are still satisfied with the solution $\frac{y^2}{2} + 4y + 6 = -\frac{e^{-9x}}{9} + \frac{x^3}{3} + \frac{1}{9}$ that appears on the screen when the V200 is used to solve $\frac{dy}{dx} = \frac{x^2 + e^{-9x}}{4 + y}$, $y(0) = -2$.

Such a problem should be accompanied by a graph of the unique solution that is announced by the existence and uniqueness theorem: students will have to solve (in the answer) for y — here it is possible — and then they will be able to plot the unique solution and see where it is defined. Plotting the slope field along with Euler's numerical method will give them a better sense of the domain of the solution.

b) We feel that the CAS calculator should be used when the method of undetermined coefficients is presented to the students. Consider the following example: a mass-spring problem governed by the differential equation $m\frac{d^2 y}{dt^2} + b\frac{dy}{dt} + k y = F_0 \cos(\omega t)$. Here, m, b, k and F_0 are positive constants and ω will be the variable. Engineering students learn that, when $b^2 < 2mk$ the amplitude of the steady state solution will reach a maximum value when $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$. The TI can be used in order to see this in a specific case. Taking, for example, $y'' + \frac{1}{4}y' + 2y = 2\cos(\omega t)$ (where ' denotes derivative with respect to t), we must find $\omega = \sqrt{\frac{2}{1} - \frac{1/16}{2}} = \frac{3\sqrt{14}}{8} \approx 1.40312$.

If students want to verify this using their symbolic calculator, they will have to apply the undermined coefficients method, solve a system of 2 equations in 2 unknowns, find the amplitude of the steady-state solution, and then find the maximum of this function (variable $\omega > 0$). Plotting also the graph of the amplitude versus the frequency will confirm (numerically) their result. The next figures illustrate these steps:
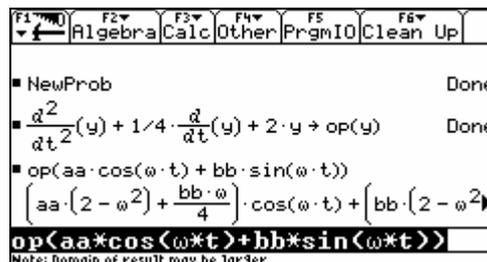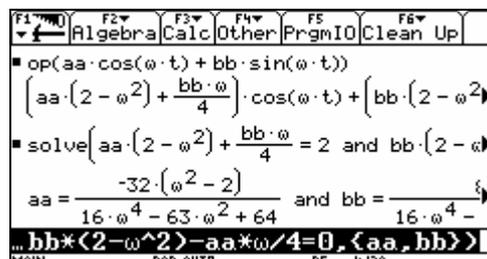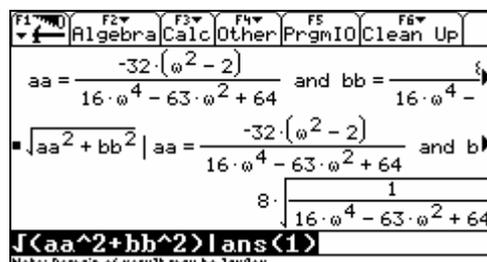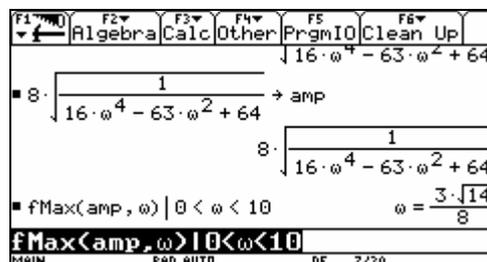

Figure 12

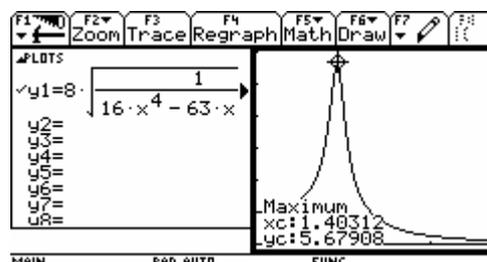Figure 13

Figure 14

Figure 15

Figure 16

We believe that this last example is a good example of "teaching mathematics in a computer algebra environment". When they apply the undetermined coefficients method, students will always have to find the correct candidates for the particular solution. The rest of the job belongs to the CAS. By the way, the V200 applies the variation of parameters method when it finds a particular solution. Thus,

if students use the V200 to directly solve the DE $y'' + \frac{1}{4}y' + 2y = 2\cos(\omega t)$, they will get a very large inconvenient answer. In fact, by the method of variation of parameters, the particular solution will be of the form $y_p = u_1 y_1 + u_2 y_2$, where

$$y_1 = e^{-t/8} \cos\left(\sqrt{127} \, t/8\right), \quad y_2 = e^{-t/8} \sin\left(\sqrt{127} \, t/8\right)$$

$$u_1 = -\int y_2 \frac{r}{W} \, dt \quad \text{and} \quad u_2 = \int y_1 \frac{r}{W} \, dt$$

Here r is the function $2\cos(\omega t)$ ─ the rhs of the ODE when the leading coefficient is 1─ and W is the wronskian: $W \equiv \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Doing all these calculations provides exactly the same answer as that obtained by the "desolve" command of the Voyage 200 (with the homogeneous solution $y_c = C_1 y_1 + C_2 y_2$ removed). Then, trigonometry is required to "simplify" the answer.

3   CONCLUSION

For both authors, using the symbolic TI calculator in our courses over the last ten years has been an exhilarating experience in a context where there is relatively little variety in courses in the engineering math curriculum. Through technology we have found new and exciting ways of presenting old and traditional math concepts. Thus, our conclusion would be that everything is fine, so let's continue. Unfortunately, this enthusiasm is limited and the examples shown, or similar ones, should be done by more colleagues. Some are not convinced that they can teach using a CAS. For them, the CAS calculator serves primarily as a graphic calculator or as an equation solver. We hope that some of our examples will convince more instructors to use CAS as a tool for teaching mathematics, without restricting its use. As far as understanding concepts and having fun in learning and doing mathematics are concerned, where is the problem?

Finally, let us give our personal answer to some colleagues who ask the following question: Will students who use a CAS be as skilled as we were 20 or 30 years ago if they don't calculate first by hand? In fact, it is probably difficult to compare two different periods considering that technology evolves so rapidly. We remember that some 20 or 30 years ago, we had to study concepts and theorems without really understanding them; we also had to do long calculations by hand, often irrelevant but necessary at that time, whose results were often false with no way to check the answer. We also know that every generation of professors finds students less skillful than they were. In fact, regular use of a CAS won't make students instant great mathematicians. However, it will enable the teacher to show students that many mathematical concepts can be best understood with technology. This is probably the point where some colleagues won't agree with us. There are those who are convinced by the advantage of using the CAS in teaching math; there are others who don't see that a CAS like the Voyage 200 can aid in the student's understanding of mathematical concepts even if the technology can be useful

in doing some specific tasks; and there are those who have decided that they won't spend much time learning a new technology if little else has changed in the curriculum.

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BIOGRAPHICAL NOTES

Michel Beaudin has been a mathematics professor at École de technologie Supérieure (ÉTS, Montréal, Canada) since 1991. He has considerable experience using DERIVE to support his teaching in calculus, differential equations, complex analysis and linear algebra. He was instrumental in the university's adoption of the TI symbolic calculator TI-92 Plus (Voyage 200). Since 1998, alone or with colleagues Gilles Picard and Kathleen Pineau, he has presented more than 45 lectures, most of them at 22 international conferences. Since 1999, Michel has been involved in ACA conferences (Education session) and has been co-chair of the TIME conferences since 2004. With Gilles and Kathleen, Michel organized two international conferences at ÉTS: TIME-2004 and ACA 2009. His main interest is in using computer algebra for teaching mathematics to engineering students.

Gilles Picard has been teaching basic mathematics courses at École de technologie supérieure for over 25 years. Gilles has been a long time user of technology in teaching mathematics and has often offered support and aid to collegues and math instructors. He has been, with Michel Beaudin and Kathleen Pineau, at the forefront of the ongoing experiment of mandatory TI symbolic calculators at ÉTS.