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Solving Second Order ODEs, Two Non-analytical Methods Revisited

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ABSTRACT

Mathematics can still be taught without using a CAS and this is probably the case in most schools and universities. Although CAS and technology are often used by instructors to demonstrate or illustrate mathematical concepts, they are rarely used by students. When we consider our mathematics curriculum, “Differential Equations” is one course that we firmly believe can and should benefit from the use of CAS. In this talk, we will report how our ODE course has evolved, as our engineering students have access to technology (Voyage 200 symbolic calculator) in the classroom at all times. This talk will show examples of what students still do by hand and what CAS allows us to do now to enrich the learning experience.

We will consider the series solutions of a second order equation with variable coefficients and numerical solution of the first order equivalent system. Many textbooks do not show the relation between these two subjects. With a CAS on every desk, we can ask students to compare results obtained with both methods. Of course, technology is a must to support this. As teachers, we still want our students to be able to do some specific computations manually. For example, they have to find the recurrence formula by hand for the coefficients of the series solution. However, we also want students to be able to compute, with some accuracy, the value of the solution at a certain point using partial sums or even graph this approximate solution. Then, converting the same equation into a first order system, they can plot the numerical generated curve obtained by the built-in RK method in the Voyage 200 or create a table of values for the approximate solution.

1. Introduction

Let us say a word about the idea behind this paper. Once someone has decided to use computer algebra in his teaching, he has to decide for what particular topics he will use it and when it is appropriate to do so. We know that computer algebra helps a lot in calculus and differential equations, but not so much in mathematical analysis, for example. Many colleagues or instructors continue to think that computer algebra is mainly useful for doing applications, not also for teaching mathematical concepts. In this paper, we want to show that the Voyage 200 symbolic calculator can be used in a very original way to solve numerically second order linear differential equations. We will explore the precision of results obtained with the “Power series” method and compare with values from a “Runge-Kutta” algorithm implemented on the Voyage 200. We will be using 2 out of 6 graphing windows of the device: the Diff Equations and the Sequence graphing modes. We will present an approach where pencil and paper techniques are still required but where technology completes the analysis.

2. Power series solutions using technology: more can be done

At ETS, the Voyage 200 calculator has been used since September 1999 but questions about solving ODEs using power series are usually formulated in the same way, year after year. Let’s look at the classic approach for this topic. Students are given a differential equation of the following type:

$$p(x)y'' + q(x)y' + r(x)y = 0, \quad y(0) = y_0, y'(0) = v_0.$$

In fact, the initial conditions are sometimes defined elsewhere than at 0, at $x = x_0$ for example, but a simple change of variables ($v = x - x_0$) will revert the problem to this general case. The functions $\frac{q(x)}{p(x)}$ and $\frac{r(x)}{p(x)}$ are supposed to be analytic at 0 and $p(0) \neq 0$ (said otherwise, 0 is an ordinary point of the differential equation). Then students are told, without proof, that there exists a general solution of the problem by mean of power series:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{with} \quad c_0 = y_0 \quad \text{and} \quad c_1 = v_0.$$

This series converge for $|x| < R$ where R is at least equal to the distance between 0 and the nearest singularities (if we suppose that p , q and r have no common factors, the nearest singularity is a complex point a which is the closest to 0 and satisfying $p(a) = 0$). Students have to find a recurrence formula in order to compute recursively the coefficients of the series. This is done by hand. Having found many coefficients, they can use them to get an approximation value of $y(\alpha)$ with α between $-R$ and R . The problem is that we never ask them how accurate is this approximation! In fact, they usually only have to estimate $y(\alpha)$ using no more than the 5 first non vanishing terms... This paper wants to reverse this approach. We will show how this subject can be enriched with the aid of technology. Moreover, because Voyage 200 “desolve” command does not possess, like

big CAS running on computers, a “power series method”, this “disadvantage” can become an advantage. A good mix of paper and pencil techniques and, also, a use of RK numerical method for first order systems can be used and give a nice solution of the problem. Let us be more concise, using an example.

3. Concrete examples

If we want to show what we are doing exactly in the classroom, the best way will be to give concrete examples. Let us take the following problems:

$$eq1: (x^2 + 4)y'' + 3xy' - 4y = 0, \quad y(0) = 4, y'(0) = 1.$$

$$eq2: (2x - 9)y'' - y' - 4xy = 0, \quad y(0) = 8, y'(0) = -2.$$

For each of them, students are asked to find a series solutions centered at 0 and use it to estimate the value of $y(\alpha)$ where α belongs to the interval of convergence. In fact, for *eq1*, we note that there are singularities located at $x = \pm 2i$, so the series $\sum_{n=0}^{\infty} c_n x^n$ ($c_0 = 4, c_1 = 1$) will certainly converge for $|x| < 2$. We will take $\alpha = 1$ in this

case. For *eq2*, the only singularity is at $9/2$, so the series $\sum_{n=0}^{\infty} c_n x^n$ ($c_0 = 8, c_1 = -2$) will converge for $|x| < 9/2$.

We will take $\alpha = 2$ in this case. One can estimate $y(\alpha)$ by using a partial sum of the form $\sum_{k=0}^n c_k$ for different values of n .

An important question is: how many terms are needed in order to be confident for the value of $y(\alpha)$? What if we only use the first 5 non vanishing terms of the series? At ETS, every student in the classroom has his own Voyage 200 on his desk. They know that the pencil and paper ability required to find the recurrence formula is still important. After recalling some properties of power series — namely that term by term differentiation is allowed within the interval of convergence —, they will set $y = \sum_{n \in \mathbb{Z}} c_n x^n$ instead $y = \sum_{n=0}^{\infty} c_n x^n$ of because we can simply decide that $c_n = 0$ when $n < 0$. This way, y' and y'' will keep the same indices, we can even omit these indices recalling that all summations are on $n \in \mathbb{Z}$ (we first saw this approach in [1] in the 80's). Substituting $y = \sum c_n x^n$ into the differential equation and, collecting similar terms, leads to the recurrence formula (because if $\sum_{n=0}^{\infty} a_n x^n = 0$ then $a_n = 0 \forall n$). Once it is obtained, they can compute some additional coefficients and, then, estimate $y(\alpha)$. Our students will manually do the substitutions and find the recurrence formula. For *eq1*, this would lead to:

Solve $(x^2 + 4)y'' + 3xy' - 4y = 0$

$$\text{Using } y = \sum c_n x^n \quad y' = \sum n c_n x^{n-1} \quad y'' = \sum n(n-1)c_n x^{n-2}$$

$$x^2 \sum n(n-1)c_n x^{n-2} + 4 \sum n(n-1)c_n x^{n-2} + 3x \sum n c_n x^{n-1} - 4 \sum c_n x^n = 0$$

$$\sum n(n-1)c_n x^n + \sum 4n(n-1)c_n x^{n-2} + \sum 3n c_n x^n + \sum -4c_n x^n = 0$$

Making the change $n \rightarrow n+2$ in the second summation so that they are all in terms of x^n

$$\sum n(n-1)c_n x^n + \sum 4(n+2)(n+1)c_{n+2} x^n + \sum 3n c_n x^n + \sum -4c_n x^n = 0$$

$$\sum [n(n-1)c_n + 4(n+2)(n+1)c_{n+2} + (3n-4)c_n] x^n = 0$$

$$\text{Thus } (n^2 - n + 3n - 4)c_n + 4(n+2)(n+1)c_{n+2} = 0$$

$$\text{and the recurrence formula will be } c_{n+2} = \frac{-(n^2 + 2n - 4)}{4(n+2)(n+1)} c_n$$

Since for $n=0$, $y(0) = 4$ and $y'(0) = 1$, using $n = 0, 1, 2, \dots$ in this formula gives us the coefficients and

$$\text{the solution } y(x) = 4 + x + 2x^2 + \frac{1}{24}x^3 - \frac{1}{6}x^4 + \dots$$

Using a partial sum with 5 terms gives us an approximation of the solution at $x = 1$, $y(1) \approx \frac{55}{8} = 6.875$

But what can be said about the precision of such an estimate? The process of calculating more terms, manually, to get more precision is quite tedious. Here is where technology can help. We show students that their Voyage 200 has a “sequence” graphing mode that can be used in order to generate as many coefficients as they need. Furthermore, the adaptive built-in RK method can find numerically the value of $y(\alpha)$ if the user has transformed the second order ODE into a first order system. So, when comes time to teach power series solutions to our students, we can do a better job if we decide to make use of technology.

We summarize in the following table the changes that have occurred for both of us for the past few years. We moved from questions a), b) and c) on the left to questions a), b), c) and d) on the right.

No technology involved	Voyage 200 involved	
a) Use a power series representation and find the recurrence formula.	a) Use a power series representation and find the recurrence formula.	
b) Find the interval of convergence of the series.	b) Find the interval of convergence of the series.	
c) Estimate the value of $y(\alpha)$ by using the 5 first non vanishing terms.	c) Find the value of $y(\alpha)$ by using as much terms as you need.	Define the partial sum $\sum_{k=0}^n c_k \alpha^k \rightarrow s(n) \quad (\alpha \text{ fixed})$ and compute the value of $y(\alpha)$ with 2 “stable” decimals.
	d) When the value of $y(\alpha)$ has become “stable”, make a second check using RK method.	By converting the second order ODE into a first order system and using the DE graphing mode window.

Figure 1 Table showing the way questions are given to the students for power series solutions of ODEs

Here are summarized the results to some of the former questions for both problems:

<i>eq1</i> ($\alpha=1$)	<i>eq2</i> ($\alpha=2$)
a) $c_0 = 4, c_1 = 1,$ $c_{n+2} = -\frac{(n^2 + 2n - 4)}{4(n+1)(n+2)} c_n \quad (n \geq 0).$	a) $c_0 = 8, c_1 = -2,$ $c_{n+1} = \frac{(2n^2 - 3n)c_n - 4c_{n-2}}{9n(n+1)} \quad (n \geq 1).$
c) $y(1) = 6.88$ (first 5 non vanishing terms). when using more terms, the answer seems to stabilize to 6.89.	c) $y(2) = 0.14$ (first 5 non vanishing terms). when using more terms, the answer seems to stabilize to 0.84.

Figure 2 Results for our two ODEs

Figure 2 indicates that if we only use the 5 first non vanishing terms, the value of $y(1)$ will be a good estimate in the case of *eq1* while the value of $y(2)$ will be very bad in the case of *eq2*. With technology, this can be easily corrected. Because of the poor estimate (0.14) obtained in *eq2*, we will show how Voyage 200 can help for this equation.

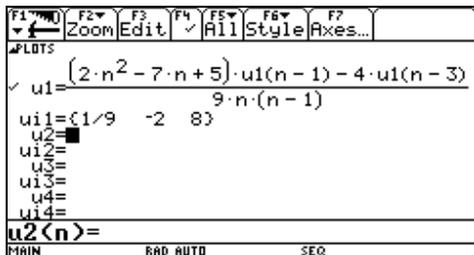
First, the Y Editor of the Voyage 200, in sequence graphing mode, requires a sequence defined in the form of “ $u1(n) =$ ” (see figure 3 a). This means that the students have to rewrite the recurrence formula using the change $n \rightarrow n - 1$, thus obtaining

$$c_n = \frac{(2(n-1)^2 - 3(n-1))c_{n-1} - 4c_{n-3}}{9(n-1)n} = \frac{(2n^2 - 7n + 5)c_{n-1} - 4c_{n-3}}{9n(n-1)}.$$

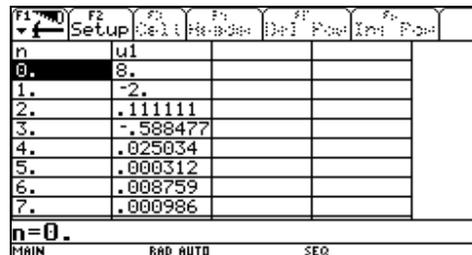
In Voyage 200, initial values should be given in reverse order (see figure 3 a) and, in this particular case where the recurrence formula is of order 3, we need to compute by hand the value of c_2 .

Recalling that $c_n = 0$ when $n < 0$, we find

$$c_2 = \frac{(2 \cdot 2^2 - 7 \cdot 2 + 5) \cdot (-2) - 4 \cdot (0)}{9 \cdot 2 \cdot 1} = \frac{2}{18} = \frac{1}{9}.$$



(a)



(b)

Figure 3 Voyage 200 Y Editor in “sequence” graphic mode (a) and (b) showing a table of values.

The coefficients of the series solution are now available through this special function $u1(n)$, as can be seen in figure 3 b. Partial sums for finding a stable value for $y(2)$ are now computed:

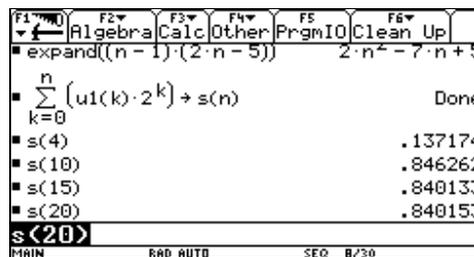


Figure 4 Function definition and evaluation

Figure 4 shows us that an appropriate 2 decimal value for $y(2)$ should be 0.84. Using only the first 5 non vanishing terms (that is $s(4)$ because c_0, c_1, \dots, c_4 are all different from 0), we would conclude that $y(2) = 0.137174\dots$

Another way of generating a partial sum for the series solution, without using the graphic environment, is to use the “when(, ,)” command which is equivalent to a “if then else “ instruction. Let’s create a conditional function $a(n)$ giving us, for $n=0,1,2,3\cdots$ the desired coefficients. For the above example, the function must give us a value of 0 if $n<0$, 8 if $n=0$, -2 if $n=1$ and $\frac{(2n^2 - 7n + 5)a_{n-1} - 4a_{n-3}}{9n(n-1)}$ if $n = 2, 3, 4, 5\cdots$. Figure 5 a below shows such a function created using four cascading when commands.

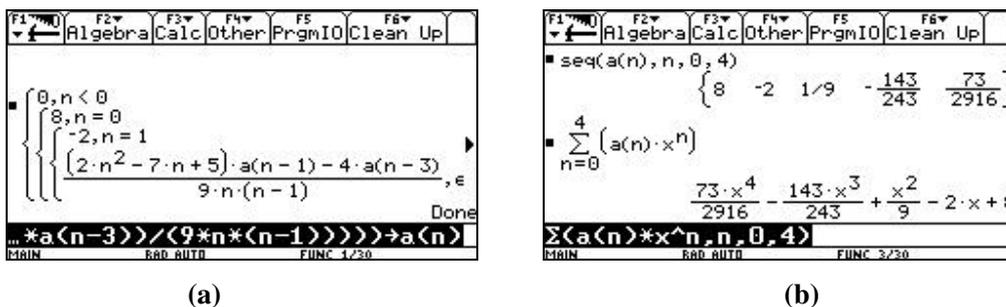


Figure 5 A function to generate the coefficients, and the partial sum of the series solution.

Looking at figure 5 b, we see one advantage of this approach. The coefficients are in exact mode instead of floating point values as in figure 3 b. On the other hand this approach is much slower if we need to calculate more precision since evaluating $a(n)$ requires recursion back to 0 for each coefficient instead of relying on previous value already calculated as with our previous $u1(n)$ function. But the user will get the same values as those obtained using the “sequence” graphic mode (see figure 6).

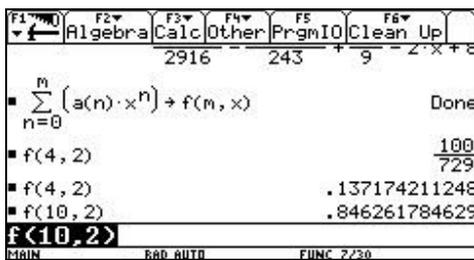


Figure 6 Estimating $y(2)$ with the $a(n)$ function.

The Voyage 200 possesses also a robust adaptive RK method [2], so students can do the same problem a second time, using another method! This is often not seen in a Differential Equations course. The Runge-Kutta methods are usually shown for first order ODE’s. To solve a second order equation we need to show students how to transform this equation in a set of first order equations. Of course, once this is done, the CAS calculator will do all the computing though students have to learn how to control the environment.

First, let's look at converting a second order ODE into a system of first order ODE. Every second order equation which can be re-written in the form $y'' = F(x, y, y')$ can be transformed in a canonical way in a system of first order equations. In figure 7 b, we see that the independent variable must be t instead of x . Furthermore, the dependant variables must be y_1, y_2, y_3, \dots . In fact, this graphic mode is designed to solve system of first order differential equations.

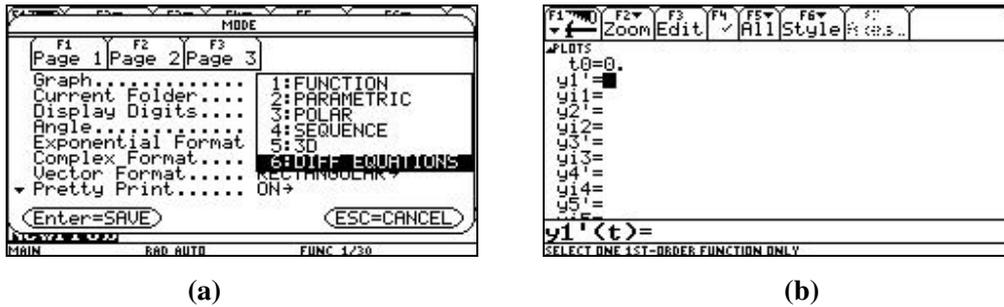


Figure 7 The “differential equations” graphing mode

Recalling that $eq2$ is $eq2:(2x-9)y'' - y' - 4xy = 0, y(0) = 8, y'(0) = -2.$, we set $x = t$, then $y = y_1$ and $y' = y_1' = y_2$, we can convert the second order ODE $eq2$ into a first order ODE system $syst_eq2$:

$$syst_eq2: \begin{cases} y_1' = y_2 & y_1(0) = 8 \\ y_2' = \frac{4ty_1 + y_2}{2t - 9} & y_2(0) = -2 \end{cases}$$

We enter these values in the Voyage 200 DE graphing mode (see figure 8 a). One can notice the particular way the initial conditions are entered ($t_0=0$ and $y_{i1}=8$ for $y_1(0)=8$). In figure 5 we see that we have chosen a graph window with x or t -axis going from -1 to 5 and the y -axis going from -5 to 10 .

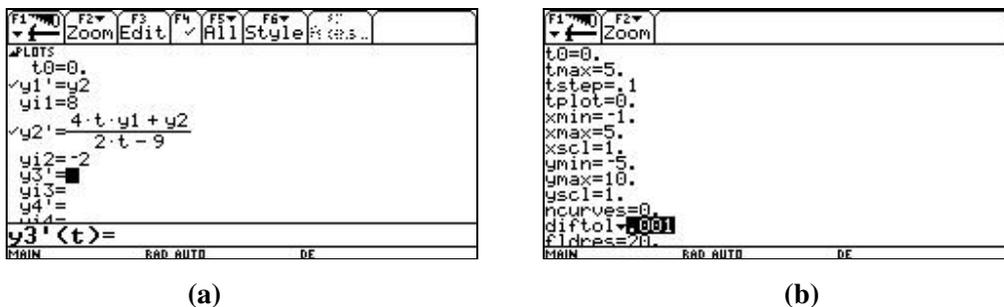


Figure 8 Preparing to estimate $y(2)$ using RK method for $syst_eq2$

The step for this example will be 0.1 though this being an adaptive step implementation of a Runge-Kutta method the step could be smaller to ensure that the global error will not exceed a value of $dif\text{tol} = 0.001$ when going from $t_0 = 0$ to $t_{\max} = 5$. In the F1 menu, one must also set the “Fields” option to “FLDOFF” to prevent the calculator from trying to plot a slope field, option we use when studying with them graphical solution of first order equations. Figure 9 shows the curve computed with this method and a table of values generated.

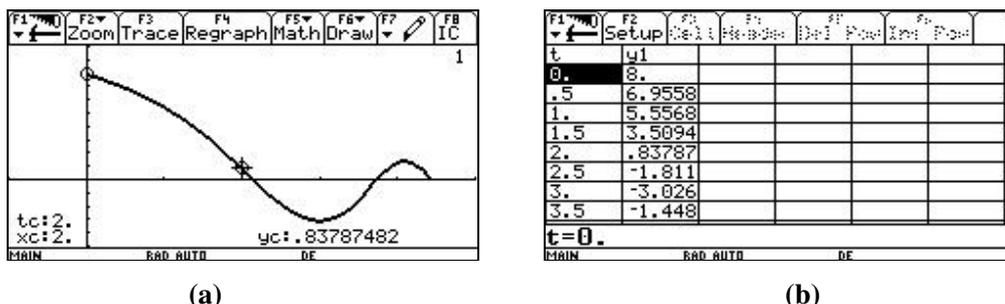


Figure 9 Solution and table of values obtained using RK method for sys_{t_eq2}

The value 0.83787 was obtained using a step size of 0.1 with a tolerance of 0.001. Changing the tolerance to 0.0001 would provide $y(2) = 0.83991$. We can be confident with the value 0.84. Note also that the graph “stops” at $x = 4.5$ even though we asked to plot a solution up to $t = 5$! This method being adaptive and wanting to ensure a global tolerance, the calculator isn’t capable of calculating a sufficiently small step, thus stopping calculations at the singularity of this equation which is $9/2$. Cool!

4. Conclusion

This approach — using a mix of paper and pencil and technology — has many advantages. One among them is the fact that students are able to “discover” their CAS calculator when they have to do this kind of problem using 2 different graphic windows. We have to admit that the RK method is only used to check the results obtained by the power series method: we don’t study RK in details (only Euler) but this can be done in a separated numerical analysis course. Another advantage of using the sequence graphing mode is contained in the following remark: students attending this ODE course had followed, earlier, a calculus course where they studied infinite series and convergence. This is a nice topic where we can talk again of these results. They know

that if a series of real numbers $\sum_{n=0}^{\infty} a_n$ converge, then we must have $\lim_{n \rightarrow \infty} a_n = 0$. The table of values in figure 3b)

shows this but, in order to evaluate a series solution of an ODE at some point, “how long will it take” if we want to be confident with a partial sum? Experimenting with the CAS Voyage 200 gave us a satisfying answer.

Some authors, especially in Applied Differential Equations, will mention that efficient numerical approximation methods and access to computer and more powerful calculators could question the need for power series method in a first course in differential equations, in particular for solutions about a regular point. Some of them will even omit this subject [3]. We still think it's a good idea to link all these mathematic topics together and of course, a partial sum of a series solution (a polynomial) is still a great object to manipulate algebraically.

5. References

- [1] Spiegel, Murray R. *Applied Differential Equations*. 3rd Edition. Prentice Hall. 1981.
- [2] P. Bogacki and L.F. Shampine. *A 3(2) Pair of Runge-Kutta Formulas*. Applied Math. Letters, vol. 2, pp. 1-9, 1989.
- [3] Brannan, James R. and Boyce, William E.. *Differential Equations, An Introduction to Modern Methods and applications*. 1st Edition. Wiley. 2007.

Also, in the writing of this article, we took a look at some exams or exercises given by our colleagues to their students. Special thanks to our colleague Chantal Trottier.