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Solving ODEs in a Larger Context

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ABSTRACT

Mathematics can still be taught without using a CAS and this is probably the case in most schools and universities. Although CAS and technology are often used by instructors to demonstrate or illustrate mathematical concepts, they are rarely used by students. When we consider our mathematics curriculum, "Differential Equations" is one course that we firmly believe can and should benefit from the use of CAS. In this talk, we will report how our ODE course has evolved, as our engineering students have access to technology (Voyage 200 symbolic calculator) in the classroom at all times. This talk will show examples of what students still do by hand and what CAS allows us to do now to enrich the learning experience.

Specific examples will focus on first order ODEs. Students not only use analytical methods, but with the aid of technology they can also focus on subjects often neglected in ODES courses. They have to plot slope fields, and in some cases also use Picard approximations and compare them, if possible, with the Taylor series expansion of the exact solution. They also have to observe where the existence and uniqueness of solutions apply. Moreover, a solution without its domain of existence can be considered incomplete.

The same approach can be applied to second order ODEs with constant coefficient and (non trivial) periodic input. Usually, textbooks introduce Fourier series for PDEs but they are very rarely used to find the steady state solution of an ODE. This is because these computations can be arduous when done manually and are more related to "applied" differential equations. With engineering students this should be part of the curriculum. Using the Voyage 200, this can be more easily explored and students should be able to plot a partial sum of the steady state solution.

1. Introduction

In an Differential Equations course where all students have access to technology in the classroom ─ this is the case at our university but is rarely seen elsewhere ─, teachers should continue to test (some of) their students ability to solve ODEs by hand but they should also go further by asking additional questions. Some of these questions should be about extra computations ─ especially those impossible by hand ─ but also questions about theory (existence of solutions for example). This is probably one aspect where technology can be used to do more mathematics, replacing some long manual calculations by more exploration of theory, concepts and applications! And you don't have to think of complex examples in order to do better mathematics. Just use your traditional stuff, but add some appropriate remarks, make some pertinent comments.

In this paper, we will focus on first order ODEs that are seen in an ordinary differential equations course, for engineering students, at university level, namely the most important ones: linear and separable. The last example will deal with a classic non-linear equation. Another talk, in a future ACA conference, will cover the second order linear ODEs. Despite all the available technology, many professors continue to teach these subjects using the same classic approach we have seen for many decades. Doing so has the following consequence: the theory of differential equations continues to be understood by many students as a course "where you have to find the good recipe for solving the problem". Fortunetaly, we now see more and more authors considering the impact of technology in this field and presenting new textbooks where we see more applications of the use of CAS and of numerical approaches. The strong relation between analysis, calculus and linear algebra should be an important part of the course but seems to be lost with all these tricks and technical methods of solutions.

2. Larger context for first order linear ODEs

We agree that students should continue to solve by hand linear and separable equations, but we also consider that questions about these could contain additional material. Nice examples can be found in [1] or [4] and our own examples will exploit some of these additional considerations. Let us recall the main difference between linear and non linear first order differential equations. The linear case has the standard form

$$\frac{dy}{dt} + p(t)y = q(t) \quad \text{with} \quad y(t_0) = y_0$$

where the functions *p* and *q* are supposed to be continuous over an open interval $I =]a\,, b[$ and $t_0 \in I$. Then, we have an *explicit* formula for the solution and this solution is also continuous over *I*. In fact, if the initial condition is dropped, then <u>all</u> solutions are included in the following formula (one has to choose *C* in order to satisfy eventually an initial condition):

$$y(t) = \frac{1}{u(t)}\left(\int u(t)q(t)\,dt + C\right) \quad \text{using integrating factor} \quad u(t) = e^{\int p(t)\,dt} .$$

And if the initial conditions are maintained, then we easily find that

$$y(t) = \frac{1}{u(t)}\left(y_0 + \int_{t_0}^{t} u(s)q(s)\,ds \right) \quad \text{where this time} \quad u(t) = \exp\left(\int_{t_0}^{t} p(w)\,dw\right).$$

In particular, if the coefficients are continuous everywhere, this unique solution is defined over the real line. And an explicit solution does not require implicit plotting if one wants to visualize the solution curve.

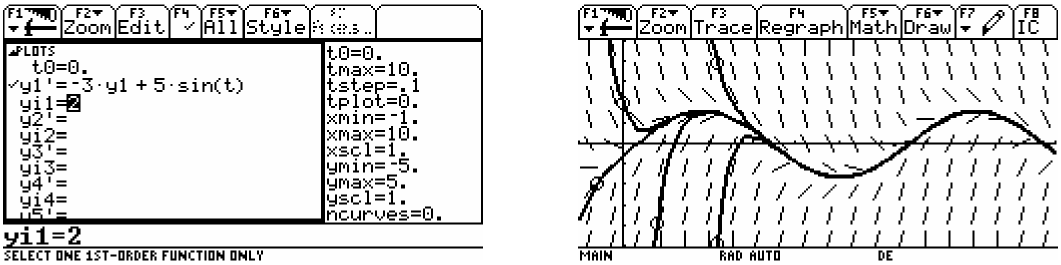
Example 1: the particular case of a first order linear DE where $p(t) = p$ is a positive constant and where the input $q(t)$ is of the form $A\sin(\omega t)$ gives the teacher a very nice opportunity to introduce the concept of steady-state solution which is important for engineering students . Let's consider such an example with

$$eq1: \quad \frac{dy}{dt} + 3y = 5\sin(t) \quad \text{with} \quad y(t_0) = y_0 .$$

Since $u(t) = e^{\int p(t)\,dt} = e^{3t}$, an integration by parts or an integrals table or a CAS is needed to compute the solution which is

$$y(t) = \alpha e^{-3t} - \frac{\cos t}{2} + \frac{3\sin t}{2} \quad \text{where} \quad \alpha = \left(y_0 + \frac{\cos t_0}{2} - \frac{3\sin t_0}{2} \right) e^{3t_0} .$$

Second, as *t* gets large and larger, the exponential terms becomes small and the solution tends to the steady-state solution, no matter what initial value is chosen for y_0. This can be illustrated using a numerical approach with technology. Here, a slope field "shows" this steady-state phenomenon. See figure 1 where the slope field is drawn in the window $-1 < t < 10$, $-5 < y < 5$, and where the curves were plotted using Euler's method with a step size of 0.1. We have used different values for y_0, with the F8 IC (initial conditions) menu:



(a) **(b)**

Figure 1 Voyage 200 DE graphing mode. Y-Editor and window parameters for Euler's method.

If the professor asks students to find the amplitude of this steady-state solution, some will take a look at the plots on figure 1(b), they will use the cursor on the Voyage 200 and find something around 1.61. The CAS has a "trigcollect" command that performs the transformation

$$-\frac{\cos t}{2}+\frac{3\sin t}{2}\rightarrow\frac{\sqrt{10}}{2}\sin\left(t-\arctan\left(1/3\right)\right).$$

The exact amplitude is $\sqrt{10}/2\approx 1.58$ This is a very simple example but a good one to illustrate what we mean by a "larger context". With technology always available, the teacher can explore with his students many avenues with a little extra works ─ which is simple with the ease of use of the Voyage 200 symbolic calculator.

Another avenue that can be explored is the relationship between Picard iterations and the symbolic solution of an ODE. We will use a simple linear equation to illustrate our point. Before going to this next example, let us recall the following result about existence and uniqueness of solution (that can be found in many textbooks). Consider the following general first order equation:

$$\frac{dy}{dt}=f(t,y)\quad\text{with}\quad y\left(t_0\right)=y_0.$$

Usually, it is assumed that the functions f and $\partial f/\partial y$ are continuous over some rectangle $R=\left[a,b\right]\times\left[c,d\right]$ containing the point $\left(t_0,y_0\right)$. Then this equation has a unique solution. More precisely, there exists a positive constant α, an interval $I=\left]t_0-\alpha,t_0+\alpha\right[\subset\left[a,b\right]$ and a unique function ϕ continuous and differentiable such that $\phi\left(t_0\right)=y_0$ and $\frac{d\phi}{dt}=f(t,\phi(t))$ for $t\in I$.

Moreover, this function ϕ can be obtained by the following method of successive approximations (Picard iterations).

We define $\quad y_0(t)=y_0,\qquad y_n(t)=y_0+\int_{t_0}^{t}f\left(s,y_{n-1}(s)\right)ds\quad\left(n=1,2,3,\cdots\right).$

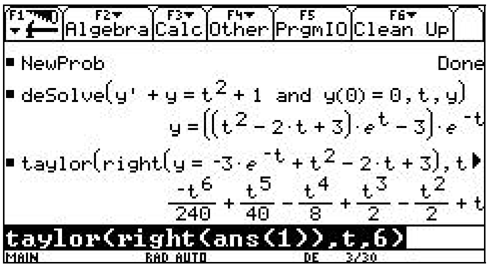
Then $\lim\limits_{n\to\infty}y_n(t)=\phi(t)$.

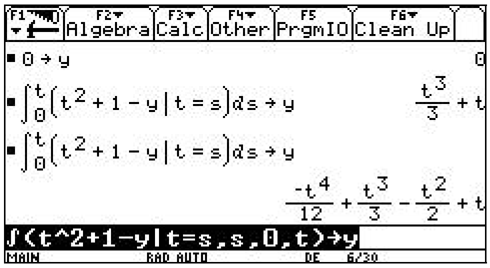
Example 2: students attending an ODE course have followed, as a prerequisite, a single variable calculus course. So they have seen the Taylor series expansion of functions. In some cases, Picard iterations can be compared to the Taylor expansion of the exact solution. Here is an example. The exact solution to

$$eq2: \quad \frac{dy}{dt} + y = t^2 + 1 \quad \text{with} \quad y(0) = 0$$

is $y = -3e^{-t} + t^2 - 2t + 3$. A "larger context" should ask students to find the Taylor expansion of this solution (quite easy here) in order to get, up to the order 6,

$$t - \frac{t^2}{2} + \frac{t^3}{2} - \frac{t^4}{8} + \frac{t^5}{40} - \frac{t^6}{240} + \cdots$$





(a) **(b)**

Figure 2 Voyage 200, solving *eq*2 then Taylor expansion. Picard iterations

Our students, with their Voyage 200 calculators can get directly the above results, see figure 2 (a). In our classrooms, the deSolve( ) command is used to verify results or to work applied problems. In this example, our students would have used the results of the previous example to get an integrating factor, do the appropriate integral and use the initial condition to get the value of the constant in the general solution. Using the Picard iterations formula, students should get successive polynomials of higher and higher degrees which should converge to the Taylor expansion of the real analytical solution. Even if there are not many first order equations where one can calculate the integrals involved in Picard iterations, this is a nice area to go back to a subject they have seen in a previous Calculus course.

Then, let's compute the successive approximations: one would find

$$y_0(t) = 0, \quad y_1(t) = \frac{t^3}{3} + t, \quad y_2(t) = -\frac{t^4}{12} + \frac{t^3}{3} - \frac{t^2}{2} + t,$$

$$y_3(t) = \frac{t^5}{60} - \frac{t^4}{12} + \frac{t^3}{2} - \frac{t^2}{2} + t, \quad y_4(t) = -\frac{t^6}{360} + \frac{t^5}{60} - \frac{t^4}{8} + \frac{t^3}{2} - \frac{t^2}{2} + t, \ldots$$

We can observe how the coefficient of a specific power term tends to the expected value. Let us indicate to the reader that our students perform these iterations on their own calculator — some will use a "Picard" function added on the CAS by teachers. Or they can implement easily the calculations on their Voyage 200 ─ see figure 1 (b). Also, if they forgot or don't want to do by hand a Taylor expansion, they will also use their calculator, as seen in figure 1 above . Of course, they need to learn the syntax of the "Taylor" function of the Voyage 200. Some 10 years ago ─ before the introduction of mandatory CAS technology in the classroom ─, these questions (in this example and in example 1 before) were not presented to students. Now, they are (probably) not as good as before when comes time to solve by hand a lot of different types of differential equations. But they have to learn how to plot a slope field (choosing an appropriate window), perform Euler's method (later on, RK method), and they have to play with Taylor expansions and Picard iterations.

3. Larger context for first order separable ODEs

Now, let's consider another aspect, a more analytical one. When students need to solve many first order ODEs, using different techniques (separable, linear, exact, integrating factors, ...), they usually don't pay attention to the domain of definition of the solution. And for a non linear equation ─ think of a separable one ─, they can obtain an implicit solution with no idea at all about where it is defined. Therefore, it is natural to change the way we want our students to solve ODEs. Here again, we will give some examples of what we mean by the expression "larger context questions".

Example 3: What happens if you ask your students the following question? According to the existence and uniqueness theorem, find the unique function ϕ which is the solution to

$$eq3: \quad \frac{dy}{dx} = \frac{1+3x^2}{3y^2-6y} \quad \text{with} \quad y(0)=1\,.$$

Most of them will solve this DE using separation of variables and they will give the following "answer": $y^3 - 3y^2 + 2 = x^3 + x$. It won't be easy for them to plot this curve because the only implicit plotting available on the Voyage 200 is the one included in the 3D window (and it takes a long time to get the graph). The graph of the solution (implicit curve), along with the point (0, 1), was generated by *Derive*, and is shown in figure 3. It gives the following information: the whole curve seems to contain 3 parts: this is not surprising since there are 3 cubic roots that can be obtained if you solve with respect to *y* the equation "answer". Unlike *Maple* where Cardano's formula would have been used, *Derive*, using Viete's formula, finds that the unique function ϕ that is solution of $eq3$ is

$$\phi(x) = 1 - 2\sin\left(\arcsin\left(\frac{x^3+x}{2}\right)\Big/3\right).$$

This kind of answer is not to be given to the students without explaining where it comes from, since they cannot find this result using their Voyage 200 ! But the following can be accomplished by the students <u>before</u> solving the ODE. First, we have

$$\frac{dy}{dx} = f(x,y) \quad \text{where} \quad f(x,y) = \frac{1+3x^2}{3y^2-6y}\,.$$

Starting from the point $(0,1)$, we see that, in accordance with the existence and uniqueness of solution theorem mentionned before example 2, if we want to find a rectangle where f and its partial derivative with respect to y are continuous, we should stay in the strip $0 < y < 2$ (shown in figure 2 also). This is not surprising because the slope becomes infinite when $y = 0$ or 2. And if we substitute the values 0 or 2 for y and solve for x in the implicit answer, we find x to be 1 or -1. Therefore, the function ϕ has the interval $]-1, 1[$ as domain.

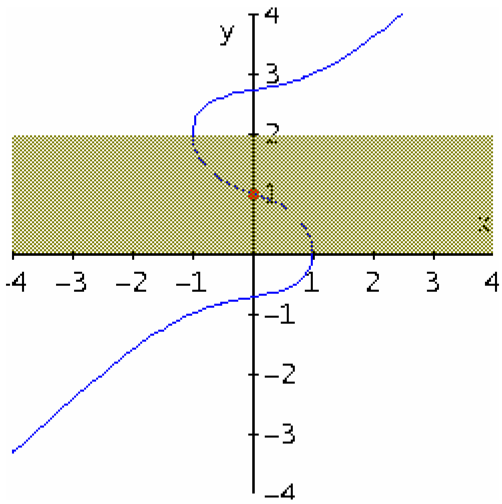
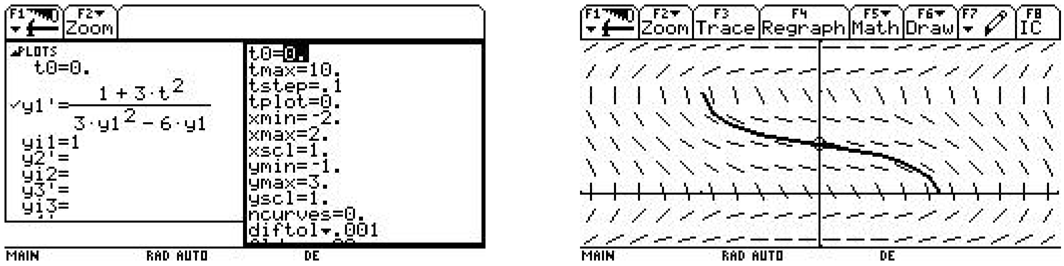


Figure 3 Implicit curve solution of *eq*3, the point $(0,1)$ and the strip $0 < y < 2$ using *Derive*

Of course, looking at the implicit plot above and considering the given initial condition, it is easy to see what part of this curve constitute the desired function solution for this problem. This example is quite different from the linear case where an explicit form for the solution is always obtained. In this last example, it was possible to expect that the solution could not be extended outside the interval $]-1,1[$. Such an example has the following advantages. Students learn the importance of a fast and robust implicit 2D plotter (the authors pointed out, in formers conferences, the importance of this functionality). They also learn that, in order to solve a cubic polynomial equation (with variable coefficients in our case), they need to learn a new formula which they would not want to use for our example! But, they can find themselves the "rectangle" R where the solution is unique and continuous, simply by solving equations with the aid of their "solve" command. So, CAS functionalities and theoretical mathematics are unified instead of being separated.

Furthermore, with the graphic DE environment on their Voyage 200, our students could solve numerically this equation as was done in our first example. Figure 4 shows the solution curve obtained with a Runge-Kutta method.



(a) **(b)**

Figure 4 Voyage 200 DE graphing mode. Parameters for Runge-Kutta method.

The Runge-Kutta method implemented in the Voyage 200 is the Bogacki-Shampine 3(2) formula, see [3]. It uses an error estimate to control step size, giving a global error not exceeding the parameter "*diftol*" when estimating the solution for the dependent variable going from *t0* to *tmax*.

For some non linear first order ODEs, nothing tells us where the solution will be defined. This is the content of our next and final example.

Example 4: $eq4: \dfrac{dy}{dx} = x^2 + y^2$ with $y(0) = 1$.

Using the "deSolve" command of the Voyage 200 calculator, we get no answer for *eq*4:

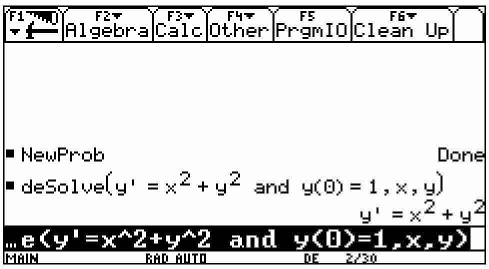


Figure 5 Voyage 200 "cannot" solve *eq*4

Students observe that their CAS calculator can't solve everything... Of course, choices made by the Texas Instruments CAS developers are motivated by the fact that the average user of the device won't need to know what is a Bessel function. *Maple*'s solution to *eq*4 is a piecewise function involving Bessel functions (we won't reproduce it). For the interval $x > 0$, the answer has a singularity near the value $x = 0.9698$. .

In order to see this, we consider the denominator of Maple's answer valid when $x > 0$. We ask Maple to solve this expression equal to zero. Figure 6 below shows the result.

$$fsolve\left(-\frac{\left(\Gamma\left(\frac{3}{4}\right)^2-\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}x^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}x^2\right)=0, x=1\right);$$

$$0.9698106539$$

Figure 6 Maple finds that *eq*4 has a singularity near $x = 1$.

How can we present this problem to our students with the assistance of their Voyage 200 calculator? There are many ways. First, ask them to plot the slope field and to try to estimate the value of $y(0.9)$. The exact answer should be near 14 but this will require a step size of 1/1000...

```
#1:   see(n) := (EULER_ODE(x^2 + y^2, x, y, 0, 1, 0.9/n, n))_(n + 1)

#2:   see(10)

#3:                                              [0.9, 5.030938065]

#4:   see(100)

#5:                                              [0.9, 11.04735043]

#6:   see(200)

#7:                                              [0.9, 12.37679080]

#8:   see(500)

#9:                                              [0.9, 13.43494639]

#10:  see(1000)

#11:                                             [0.9, 13.85015525]
```

Figure 7 *Derive*'s built-in Euler's method showing values of y (0.9) for different step sizes

In fact, with the Voyage 200, a step size of 0.1 yields the value $y(0.9) = 4.8$ ... but the good point is that RK method, with its adaptive step size (using the default value *diftol* = 0.001 as global precision parameter), gives the correct answer $y(0.9) = 14.1$ and will give an "undef" answer if you try to estimate y(1), thus refusing to "cross" the singularity. There is nothing, in the DE *eq*4 that tells us about some danger near $x = 1$. Technology gives an indication only if we think to investigate near the point x = 1. Because we don't have to apply Euler's method by hand and we do have access to technology in the classroom, more time can be spent on mathematics. Following [2], here is an interesting mathematical approach to discover the singularity near x = 1. Instead of trying to solve *eq*4, replace it by these ones:

$$eq4\text{a}: \frac{dy_a}{dx} = {y_a}^2, y_a(0) = 1; \qquad eq4\text{b}: \frac{dy_b}{dx} = 1 + {y_b}^2, y_b(0) = 1.$$

It is easy to solve each of these separable ODEs and we find

$$y_a(x) = \frac{1}{1-x} \quad \text{for} \quad -\infty < x < 1, \qquad y_b(x) = \tan\left(x + \frac{\pi}{4}\right) \quad \text{for} \quad -\frac{3\pi}{4} < x < \frac{\pi}{4}.$$

Note that $y_a(x) = \frac{1}{1-x}$ has a singularity at x = 1 and that $y_b(x) = \tan\left(x + \frac{\pi}{4}\right)$ has a singularity at $x = \frac{\pi}{4}$. Note also the obvious fact when $0 \le x \le 1$: $y^2 \le x^2 + y^2 \le 1 + y^2$. So, the slopes of the curves of *eq*4b are greater than the ones of *eq*4a and, consequently, eq4 must have a singularity located between $\frac{\pi}{4}$ and 1. This is known as the "comparison theorem".

4. Conclusion

Trying to solve ODEs in a "larger context" gives us, as teachers, the freedom to explore our old stuff with a new eye. For our students, this tells them that mathematics ─ and differential equations is one of the best examples ─ can be more attractive if computer algebra supports its teaching and its learning. Of course, to do so, the teacher must be willing to use technology in his daily teaching. And this require for him to learn how to use it. It takes some time (but gives a lot of fun!).

5. References

[1] Boyce, William E. and Diprima, Richard C.. *Elementary Differential Equations and Boundary Value Problems.* 9th Edition. Wiley. 2009.

[2] Kostelich, Eric J. & Dieter Armbruster. *Introductory Differential Equations, From Linearity to Chaos*. Addison-Wesley, 1997.

[3] Bogacki, P. and Shampine, L.F. *A 3(2) Pair of Runge-Kutta Formulas.* Applied Math. Letters, vol. 2, pp. 1-9, 1989.

[4] Brannan, James R. and Boyce, William E.. *Differential Equations, An Introduction to Modern Methods and applications*. 1st Edition. Wiley. 2007.