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Laplace Transforms, ODEs and CAS

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ABSTRACT

Mathematics can still be taught without using a CAS and this is probably the case in most schools and universities. Although CAS and technology are often used by instructors to demonstrate or illustrate mathematical concepts, they are rarely used by students. When we consider our mathematics curriculum, "Differential Equations" is one course that we firmly believe can and should benefit from the use of CAS. In this talk, we will report how our ODE course has evolved, as our engineering students have access to technology (Voyage 200 symbolic calculator) in the classroom at all times. This talk will show examples of what students still do by hand and what CAS allows us to do now to enrich the learning experience.

Laplace transform in differential equations is a great subject where CAS can be used in an efficient manner. Starting with the classic definition of the Laplace transform, we already have a nice area to use our CAS, reviewing with students when an improper integral converges. In engineering, we often have to work with piecewise continuous functions, so we introduce the unit step (Heaviside) function and show them how to create it on their Voyage 200 so they can easily plot these type of functions. We still demand that our students learn and work with a basic Laplace transforms table. We have them work, by hand, many of the classic properties of Laplace transform. When comes time to calculate inverse Laplace transforms, we let them use an "Expand" command on their Voyage 200 so they can get directly the result of the partial fraction expansion of rational functions. This being done, they still have to finish manually the inverse process, going back to their table and doing, when needed, the completion of the square for complex roots or using the appropriate property of the table (time-shifting for example). As for solving differential equations with the aid of Laplace transforms, we will show an example where, even with a discontinuous forcing function, the solution can still easily be found with the aid of the CAS and we insist on having them plot this solution. We will complete this presentation by solving a system of differential equations using the Laplace transform method, which is rarely done due to the amount of manual calculations involved.

1. Introduction

A first course in Differential Equations is a great topic where Computer Algebra System (CAS) can have a significant impact on exploring solutions, on shortening part of the often long and tedious calculations related to some of the classic techniques usually covered in this course. Our students have all followed in a previous semester a Single-variable Calculus course where they have learned to calculate derivatives and integrals, by hand, with the aid of basic tables and techniques. This was shown to them even if each of them had on his desk a Voyage 200 CAS calculator. They understand that even if they will be able to use technology later in their work, they need to understand and work manually all the basic math topic in their curriculum. Then, when problems become more complex, as often the case in applied fields, they are allowed the use of CAS in solving problems. The same is true in this course, we will teach them the classic methods of solutions of differential equations, but with a twist: they are allowed to use their CAS calculators to calculate derivatives, integrals and to solve equations or systems of equations (considering that this has been learned in earlier courses). The details of the solutions must be given manually and we must always see clearly what was done with the CAS. Of course, by the time we get to Laplace transform after the mid-terms, they realize that learning to work efficiently with technology demands a lot of practice and experimentation. This paper will show what is still done by hand and where technology can have an impact on the learning experience.

2. Laplace transform, definitions and basic techniques

Let's recall the basic definition of the Laplace transform of a function $f(t)$ defined when $t \geq 0$.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt \text{ where } s \text{ is a complex variable (in our case, we'll say a real variable)}$$

The Laplace transform is said to exist whenever this integral converges. This is a nice occasion to recall improper integrals seen in a Calculus course. When asked if this definition will be difficult for them to apply, students recall that they are allowed to use the Voyage 200 to evaluate integrals so they think this won't be difficult.

Let's look at what happens with the Voyage 200 when evaluating $\mathcal{L}\{1\}$, $\mathcal{L}\{e^{-2t}\}$ and $\mathcal{L}\{\cos(\omega t)\}$.

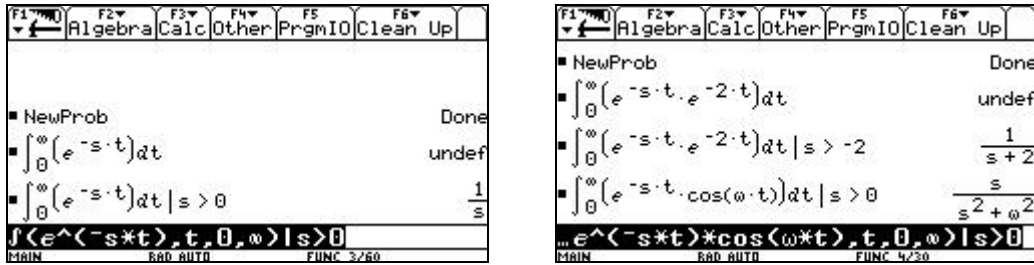


Figure 1 Computing Laplace transform for $f(t)=1$, $f(t)=e^{-2t}$ and $f(t)=\cos(\omega t)$

This is a good introduction for a discussion on conditions for existence of Laplace transform. With the aid of the Voyage 200 and showing them different properties and theorems related to this definition, we construct with them a basic Laplace transforms table, part of this table is shown below.

	$f(t)$	$F(s)$
P1	1 ou $u(t)$	$\frac{1}{s}$
P2	t	$\frac{1}{s^2}$
P4	e^{-at}	$\frac{1}{s+a}$
P7	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
P8	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
P14 a	$\delta(t-a)$	e^{-as}
P16	$f'(t)$	$sF(s) - f(0^+)$
P17	$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$
P19	$e^{-at} f(t)$	$F(s+a)$
P22	$f(t-a)u(t-a)$	$e^{-as} F(s)$
P26	$g(t)u(t-a)$	$e^{-as} \mathcal{L}\{g(t+a)\}$

Table 1 Part of our Laplace transform table

As was the case when learning how to do integrals, they are asked to use this table to calculate Laplace transforms and inverse Laplace transforms. It is important, especially in engineering, to be able to deal with discontinuous functions (which we will consider to be piecewise continuous and of exponential order so we know that the transform exists).

Example 1: a force grows linearly from 0 to 10 Newtons in 5 seconds, then is constant for the next 5 seconds and finally dies out at $t=10$ seconds. We show students how to write this in terms of the classic unit step function, often called Heaviside function:

$$H(t-a) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases} \text{ which has a jump discontinuity at } t = a. \text{ Its Laplace transform is easily}$$

found, $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$. Explaining to students that this function can be viewed as a switch to turn on or off

functions, the above example can be re-written as $f(t) = \begin{cases} 2t & 0 \leq t < 5 \\ 10 & 5 \leq t < 10 \\ 0 & 10 < t \end{cases} = 2t + (10 - 2t)u(t-5) - 10u(t-10).$

The unit step function is not implemented on the Voyage 200, but we do have access to a classic sign function, so we can define with them our own unit step function, and what is even more important is to be able to graph multiple part functions or functions with jump discontinuities (see figure 2). They can also use properties, like P22 or P26 above, to calculate the Laplace transform of such functions. In this example, we would get

$$\mathcal{L}\{f(t)\} = F(s) = \frac{2}{s^2} - 2\frac{e^{-5s}}{s^2} - 10\frac{e^{-10s}}{s} \text{ using P2, P22 and P26 from the above table of Laplace transform.}$$

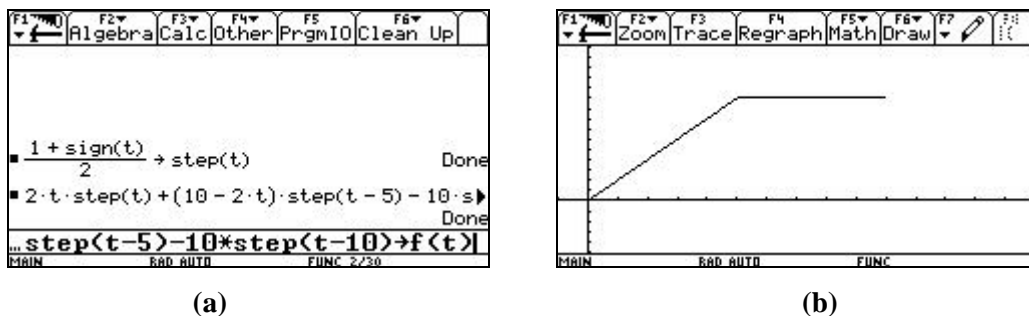


Figure 2 Creating a unit step function and graph of function $f(t)$ of the above example.

3. Inverse Laplace Transform and solving differential equations.

One important fact about Laplace transforms, when regarding applied differential equations (linear with constant coefficients) is that we can move from solving techniques heavily dependant on derivatives and integrals (undetermined coefficients, variation of parameters) to solving algebraic equations in the s -domain via the transformation done. Then, taking the inverse Laplace transform for the solution brings us to the desired solution in the time-domain. When doing this, students must always revert to their table but the Voyage 200 can simplify some of the calculations. One the most time-consuming case is the use of partial fraction expansion. Our students can use a specific command on their calculator that does this but they have after that to finish manually the problem. In some cases there is not much left to do but to read directly in their table the correct functions. On other occasions, they could need to complete the square on second order polynomials (with complex roots) or apply some other properties (in case of time shifting for example). You will find below, some examples

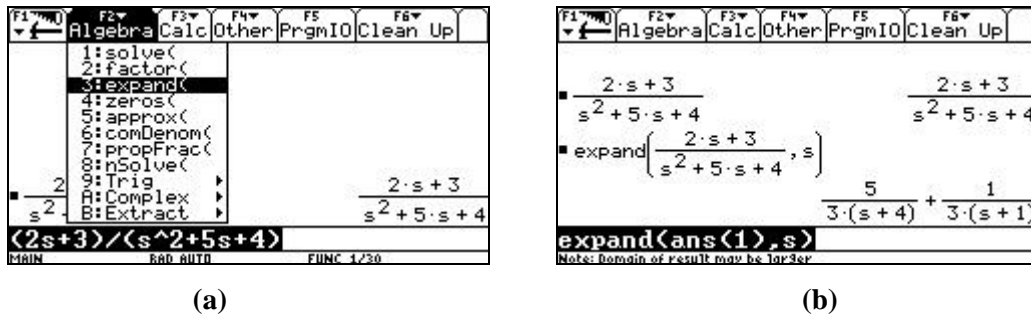


Figure 3 Examples of simplifications done by the Voyage 200

Example 2: in figure 3, we see a simple example where the “expand” command of the Voyage 200 does the partial fraction expansion. Of course, in a simple case like this one, we also show students how to do it manually.

Then, with their table, they can get the desired response for the inverse, $\frac{5}{3}e^{-4t} + \frac{1}{3}e^{-t}$. Of course, as is the case in all textbooks, we are cautious with the values chosen, even if the CAS does the job, see figure 4 a).

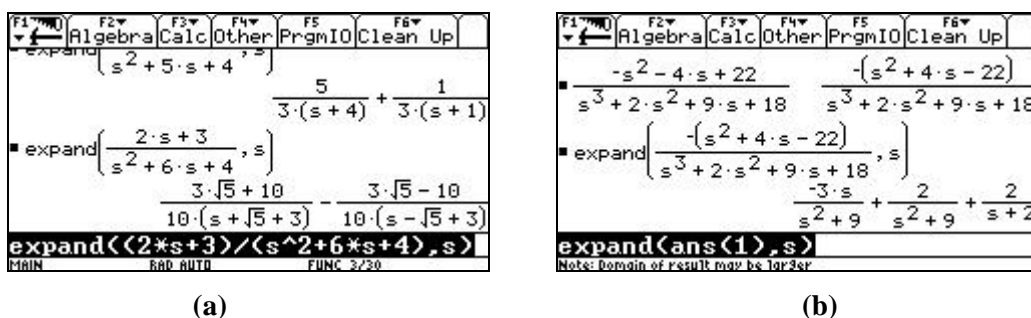


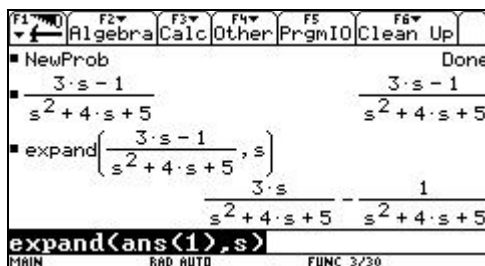
Figure 4 Examples of simplifications done by the Voyage 200

Technology is especially useful for more complex rational fractions, as in figure 4 b). Once the decomposition is done, students can get directly the answer from their table, $-3\cos(3t) + \frac{2}{3}\sin(3t) + 2e^{-2t}$. But having a “expand” command doesn’t solve all the work that they must do.

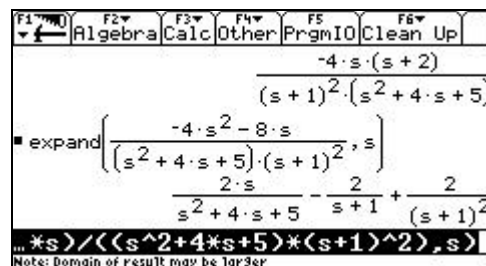
Example 3: looking at figure 5 a), we see that the calculator doesn’t really do anything, besides splitting the fraction, which in this case is a bad idea (the problem will be longer to solve in this way). This is caused by the complex roots of the polynomial $s^2 + 4s + 5$. Students must manually complete the square of this polynomial in order to use the adequate formula in their Laplace transform table (formula like P8 in table 1).

This will mean the following transformations:

$$\frac{3s-1}{s^2+4s+5} = \frac{3s-1}{(s+2)^2+1} = \frac{3(s+2)-7}{(s+2)^2+1} = \frac{3(s+2)}{(s+2)^2+1} - 7 \frac{1}{(s+2)^2+1} \text{ giving } 3e^{-2t} \cos(t) - 7e^{-2t} \sin(t).$$



(a)



(b)

Figure 5 Examples of simplifications done by the Voyage 200

Combining the previous remarks and examples shows that allowing students to use their Voyage 200 for partial fraction expansion enables them to easily do a problem like this one, $\mathcal{L}^{-1} \left\{ \frac{-4s^2 - 8s}{(s^2 + 4s + 5)(s + 1)^2} \right\}$, while still having to do part of the solution manually. Students having to do manually the decomposition will spend much more time resolving this problem than those who can start with the result in figure 5 b). They also have, in some cases, to use properties of Laplace transforms.

Example 4: to solve this simple problem, $\mathcal{L}^{-1} \left\{ \frac{3e^{-\pi s}}{s^2 + 9} \right\}$, they need P22 in table 1: with $F(s) = \frac{3}{s^2 + 9}$ and $f(t) = \sin(3t)$, then we will get $\sin(3t)|_{t \rightarrow t - \pi} u(t - \pi) = \sin(3t - 3\pi)u(t - \pi) = -\sin(3t)u(t - \pi)$. They also have to

be able to graph this solution, which as can be seen in figure 6 b) is clearly the function $\sin(3t)$ (for $t \geq 0$) shifted π units to the right. The graph shown is for t going from $-\pi$ to 4π , while the y-axis goes from -2 to 2 .

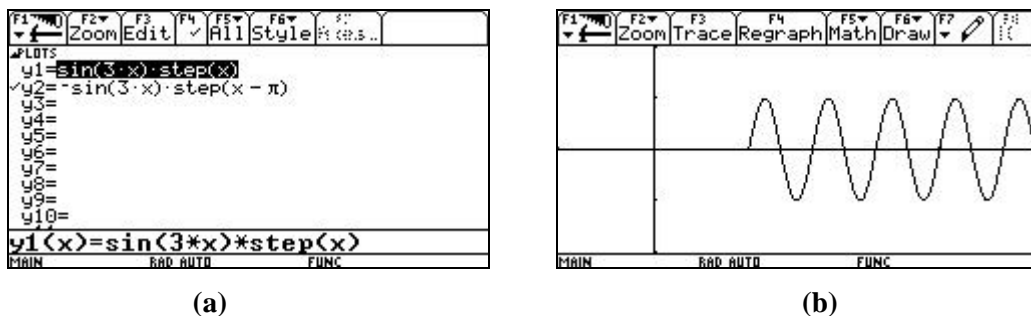


Figure 6 Examples of time-shifting and graph with the Voyage 200

Using P16 and P17 and other formulas in table 1, it is easy to transform linear differential equations with constant coefficient into algebraic equations in the s-domain.

Example 5: using the notation $\mathcal{L}\{x(t)\} = X(s)$, the differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 18x = u(t-2)$ with

$x(0) = 2$ and $x'(0) = -13$ can be transformed into $X(s^2 + 6s + 18) = 2s - 1 + \frac{e^{-2s}}{s}$. Students have to learn to be

efficient when using these special commands on their Voyage 200. If they solve this last equation for X with their Voyage 200, they get a correct but not useful form of the solution (see figure 7a). Doing a partial fraction expansion with the “expand” command on this last answer doesn’t give the best approach (see figure 7b) since they will have to complete the square to use their table of Laplace transforms (the exponential in the denominator isn’t very useful neither). Curiously, applying the “PropFrac” command to the result will give a much better form for transforming the answer back in the time-domain (see figure 8)

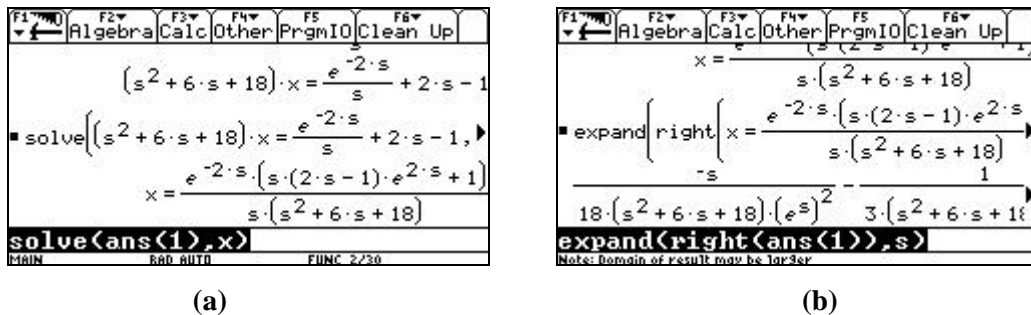


Figure 7 Examples of simplifications done by the Voyage 200

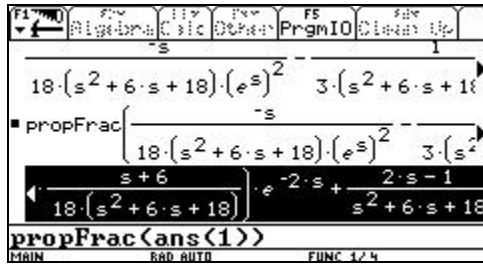


Figure 8 Simplifying with the “propFrac” command.

Students are shown that in this case, without solving the equation with the Voyage 200, they should manually see that the solution is $\frac{2s-1}{s^2+6s+18} + e^{-2s} \left(\frac{1}{s(s^2+6s+18)} \right)$. They could then “expand” the last part and complete manually the inverse process which would get them at the same point as what is shown in figure 8.

Let’s look now at a more applied problem.

Example 6: a mass-spring system leads to the following differential equation:

$$2 \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 34x = 8\delta(t) + 8\delta(t-2) - 8\delta(t-3), \text{ where } \delta \text{ is the Dirac delta function (or impulse function).}$$

Considering that the initial conditions (position and velocity) are 0, that is $x(0) = x'(0) = 0$, students manually transform this equation into $2s^2X + 4sX + 34X = 8 + 8e^{-2s} - 8e^{-3s}$, which is easy to solve for X , again by hand,

$$X = \frac{4}{s^2 + 2s + 17} + \frac{4e^{-2s}}{s^2 + 2s + 17} - \frac{4e^{-3s}}{s^2 + 2s + 17}.$$

To inverse this, students have to notice the similarity between the 3 terms, and complete the square:

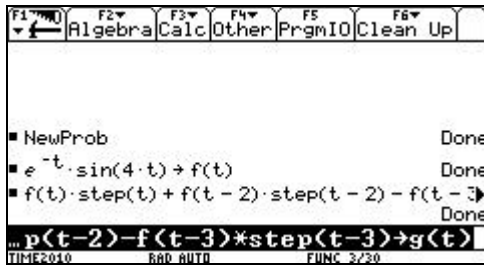
$$X = \frac{4}{(s+1)^2 + 16} + \frac{4}{(s+1)^2 + 16} e^{-2s} - \frac{4}{(s+1)^2 + 16} e^{-3s}$$

Using P8, they put $f(t) = \mathcal{L}^{-1} \left\{ \frac{4}{(s+1)^2 + 16} \right\} = e^{-t} \sin(4t)$.

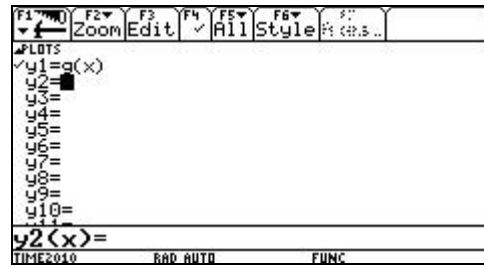
And finally, according to P22 (because of the exponentials), they obtain the solution:

$$\begin{aligned} x(t) &= f(t) + f(t-2)u(t-2) - f(t-3)u(t-3) \\ &= e^{-t} \sin(4t) + e^{-(t-2)} \sin(4t-8)u(t-2) - e^{-(t-3)} \sin(4t-12)u(t-3) \end{aligned}$$

Students were asked to graph the solution to this harmonic motion problem and to find when this mass was the furthest away of the equilibrium point and to get this maximum distance. Figures 9 and 10 below show the work done by the Voyage 200.

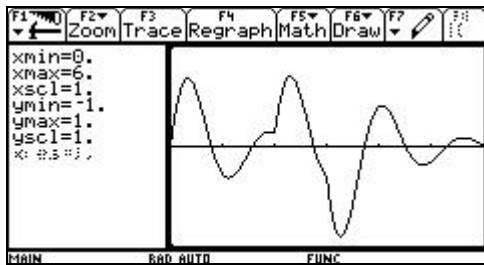


(a)

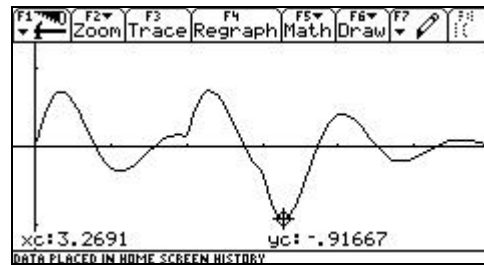


(b)

Figure 9 Entering the solution in the Voyage 200



(a)



(b)

Figure 10 Graph of the solution with the desired extremum

One can clearly see in figure 10 the impacts of the Dirac functions at $t = 2$ and $t = 3$ seconds. Figure 10b) shows that the mass is the furthest away of equilibrium at $t \approx 3.27$ seconds.

The last example concerns solving, by Laplace transform, a system of linear (constant coefficients) differential equations. This topic is often not seen or neglected considering the long calculations involved when done completely by hand. We will see that with the help of the Voyage 200 for solving the equivalent system in the s -domain and for partial fraction expansion of resulting rational fractions obtained, this type of problem can easily be done by students.

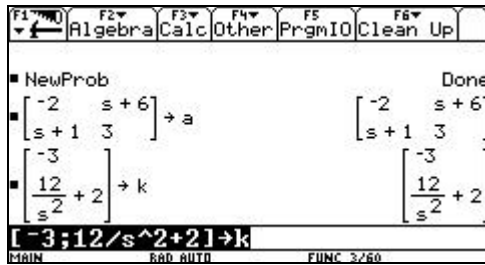
Example 6: let's consider the following system of differential equations:

$$\begin{cases} \frac{dy}{dt} + 6y = 2x \\ \frac{dx}{dt} + x + 3y = 12t \end{cases}, \text{ with } x(0) = 2 \text{ et } y(0) = -3$$

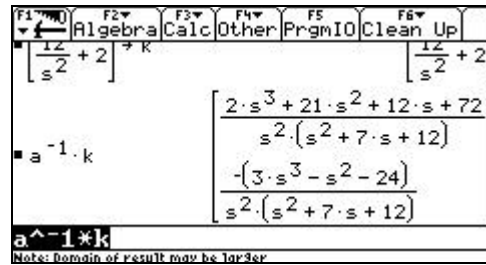
As previously, students manually take the Laplace transform of each differential equation, using their Laplace transforms table:

$$\begin{aligned} -2X + sY + 3 + 6Y &= 0 \\ sX - 2 + X + 3Y &= \frac{12}{s^2} \end{aligned} \Rightarrow \begin{bmatrix} -2 & s+6 \\ s+1 & 3 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{12}{s^2} + 2 \end{bmatrix}$$

Let's store $\begin{bmatrix} -2 & s+6 \\ s+1 & 3 \end{bmatrix}$ in A and $\begin{bmatrix} -3 \\ \frac{12}{s^2} + 2 \end{bmatrix}$ in K , so that we will have $A \cdot \begin{bmatrix} X \\ Y \end{bmatrix} = K \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = A^{-1}K$



(a)



(b)

Figure 11 Solving the system with matrices

This gives us $X = \frac{2s^3 + 21s^2 + 12s + 72}{s^2(s^2 + 7s + 12)}$ and $Y = \frac{-(3s^3 - s^2 - 24)}{s^2(s^2 + 7s + 12)}$.

Using the command « Expand » to compute partial fractions (see figure 12), our students obtain that

$$X(s) = \frac{-29}{2(s+4)} + \frac{19}{s+3} - \frac{5}{2s} + \frac{6}{s^2}, \text{ and } Y(s) = \frac{-29}{2(s+4)} + \frac{38}{3(s+3)} - \frac{7}{6s} + \frac{2}{s^2}$$

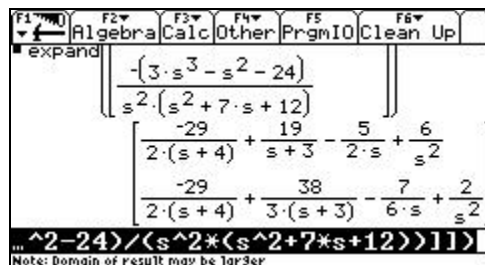


Figure 12 Partial fraction expansion

which easily gives $x(t) = \frac{-29}{2}e^{-4t} + 19e^{-3t} - \frac{5}{2} + 6t$ and $y(t) = \frac{-29}{2}e^{-4t} + \frac{38}{3}e^{-3t} - \frac{7}{6} + 2t$.

4. Conclusion

In this lecture we tried to show how we balance the equation around the big debate on basic skills and technological skills. We still show our students how to manually do basic problems. They always have to go back to their basic Laplace transforms table. But many calculations are best done using technology. This mix enables us to do more exploration with students, keeps them more focused (since we are playing with technology) and shows them where CAS answers come from. In exams, we can give them problems often more complex than what traditional students would get.

We give them access to a series of functions developed by Lars Frederiksen (originally for the TI-89) for them to verify their answers doing problems with Laplace transforms. With these, they can directly obtain the Laplace transforms and inverse Laplace transforms as well as solve differential equations (or systems) using Laplace transforms. This will be useful to many of them who will work heavily with Laplace transforms (electrical engineering for example). Looking back at our examples 4 and 5, we see in figure 13 below direct answers obtained with these special functions, although in figure a) the format of the answer differs quite a bit from what would be obtained using properties of the table of Laplace transforms.

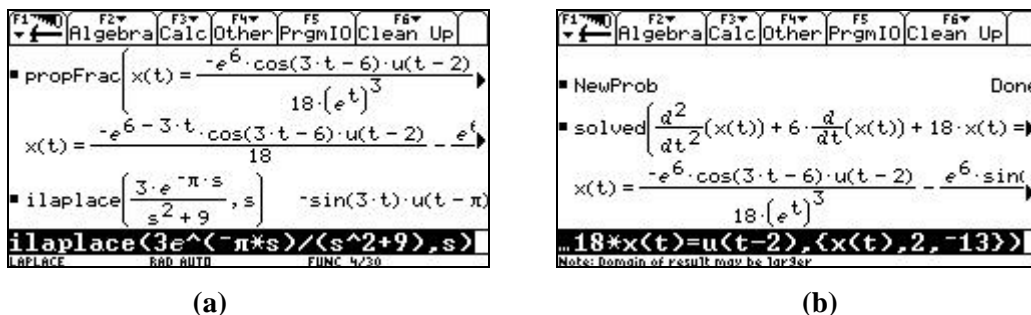


Figure 13 Direct answers to examples 4 and 5

We would not want an approach without technology, but we do not want a black box approach where students just change a few numbers in a file prepared by the teachers, not knowing exactly what the CAS will do. It would be nice for example to have a unit step function on the Voyage 200, but showing students how to create their own Heaviside function is also interesting, giving them the occasion to better understand the concepts we are teaching to them.