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Revisiting surprising results with CAS calculators

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ABSTRACT

We teach a variety of math topics (review of College Algebra, Calculus, Differential Equations, applied probability and statistics...) in a Technical Engineering School. The Voyage 200 (or TI 89 Titanium) is mandatory for all new full-time students. We make use of this calculator on a regular basis, for exploring with students, in the classroom, all the classical curriculum in mathematics.

In July 2006, in Dresden, using the Voyage 200 handheld calculator, equipped with the operating system OS 3.10, we showed some surprising results given by the device. We showed examples where the CAS system gave unsatisfactory or strange results or where it couldn't perform some commands. These problems were often related to the way the CAS system would simplify (or not) some expressions in some intermediate step of the calculations. We made some suggestions in order to fix the encountered bugs. We are happy to see that in Nspire CAS, OS 1.4, some of these bugs have been fixed and, in this talk, we will use Nspire CAS (both handheld and software) to show it and to see which bugs still remain. After revisiting these old examples, we will show some new problems encountered with our students while using the Voyage 200 CAS engine. These surprising results are still not resolved as of Version 1.4 (2008-07-09) of Nspire CAS.

1. Introduction

In order to quickly have a portrait of the actual situation, we have recalled the examples from the Dresden symposium and showed, using tables, which problems are fixed and which issues are still not resolved. In the last section, we give examples of new problems, encountered in the last 2 years.

2. Problems involving simplifications

Let us take a look at the following table:

The problem with Voyage 200 OS 3.10	Is the problem fixed with Nspire CAS handheld OS 1.4.11643?	Is the problem fixed with Nspire CAS computer software OS 1.4.11654?
<p>Example 2.1: A memory problem appears if we want to compute the first derivative of</p> $x\sqrt{25-x^2} + \sqrt{25-x^2}\sqrt{x^2+24} \rightarrow a(x)$	<p>No!!! We get the message: “Error: Resource exhaustion”</p>	<p>No!!! We get the message: “Error: Resource exhaustion”</p>
<p>Example 2.2: if we try to compute the fourth order Taylor polynomial of the expression $f = \sqrt{1+x^3}$, a “memory error” message appears.</p>	<p>Yes: because a new “generalized series function” is applied instead of computing Taylor expansion. But the CAS cannot compute the fourth derivative of f.</p>	<p>Yes: because a new “generalized series function” is applied instead of computing Taylor expansion. But the CAS cannot compute the fourth derivative of f.</p>
<p>Example 2.3: $\lim_{t \rightarrow \infty} -4(2 \cos t + \sin t)e^{-t/2} + 8$ does not simplify.</p>	<p>Yes, simplifies to 8.</p>	<p>Yes, simplifies to 8.</p>
<p>Example 2.4: $\int_0^{2-x} \int_{x^3-x} (x^3 y^4 + xy^2) dy dx$ does not simplify, except if we expand the integrand first.</p> <p>$\iint_D (-7x^6 y^6 - 5x^4 y^4) dA, D$ the unit disk does not simplify, except if we expand the integrand first.</p>	<p>Yes for both.</p> <p>For the second one, calculation in exact mode takes a little over 35 seconds. If we expand the integrand, the response is almost immediate, as it was the case with the Voyage 200. So this last problem is solved perhaps with more processing power, not with a better algorithm.</p>	<p>Yes for both.</p> <p>The second example, in exact mode, is solved in less than 1 second. We see here a good example of the difference in processing speed between the handheld and the computer version of Nspire.</p>
<p>Example 2.5: solving $d^4 - 20d^2 + 4 = 0$ with respect to d is not fully simplified. For example, one root is given as $\sqrt{-2(2\sqrt{6}-5)}$ instead of being written $\sqrt{10-4\sqrt{6}} = \sqrt{6}-2$ which is easier for students.</p>	<p>No,</p> <p>As mentioned in our 2006 paper, using the “expand()” command can simplify the result to a simpler form.</p>	<p>No</p>

Table 1

So, as we can see, 3 out of 5 problems have been fixed, this is good. But example 2.1 still causes problem for both the software and the handheld (see figure 1a):

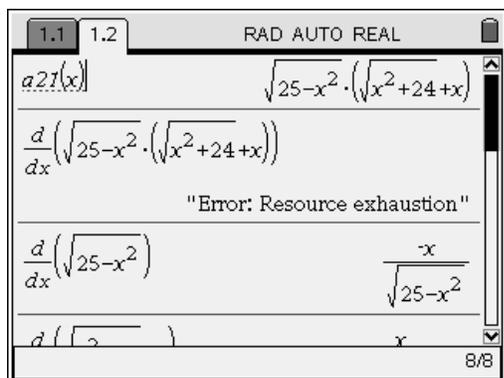


Figure 1a

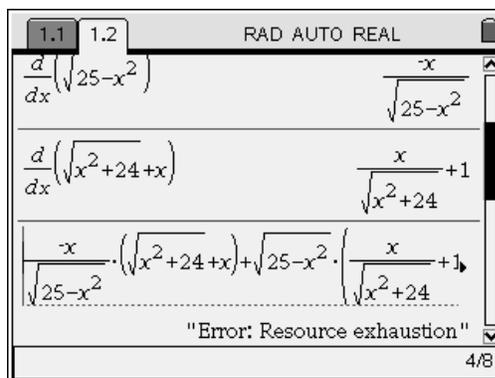


Figure 1b

Nspire simplifies the original expression $x\sqrt{25-x^2} + \sqrt{25-x^2}\sqrt{x^2+24}$ to a compact form $\sqrt{25-x^2}(\sqrt{x^2+24}+x)$ but still cannot calculate its derivative. As with the Voyage 200, the problem is not with applying derivative rules but with the sum of fractional terms involving radicals. In figure 1b, we see the same problem even if we apply manually the product rule for derivatives.

As for example 2.5, we still believe that the CAS system should simplify an expression like $-2(2\sqrt{6}-5)$ into the simpler form $10-4\sqrt{6}$. The calculator could then obtain the desired simplified form of the root by taking the square root of this last expression (see figure 2). As we mentioned in our 2006 paper, simple expression involving integers and radicals should be, if possible, simplified to the general form $a+b\sqrt{c}$, where $a, b \in \mathbb{Z}$ and $c \in \mathbb{N}$.

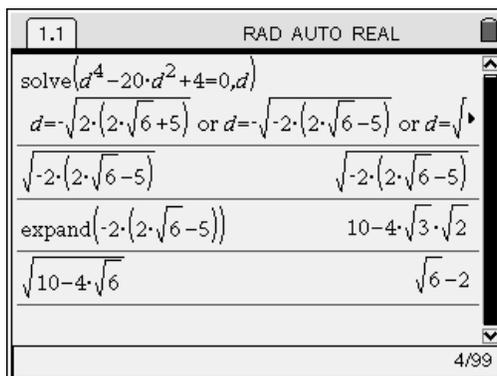


Figure 2

3. Problems with integrals

The next table shows that only example 3.3 was fixed. Let us recall that the problem in this example came from a wrong nested radicals simplification (see, after the table, the comparison between Voyage 200 screen and Nspire CAS handheld screen).

The problem with Voyage 200 OS 3.10	Is the problem fixed with Nspire CAS handheld OS 1.4.11643?	Is the problem fixed with Nspire CAS computer software OS 1.4.11654?
Example 3.1: $\int_1^2 \sqrt{\frac{1}{x^2-1}} dx$ gives only a floating point answer.	No. Only a floating point answer.	No. Only a floating point answer.
Example 3.2: $\int e^{x^3+x}(3x^2+1)dx$ is not simplified.	No.	No.
Example 3.3: $V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} dz dy dx$ simplifies to a wrong floating point answer because of a wrong nested radicals simplification!	Yes!	Yes!

Table 2

In figure 3a, the wrong answer is given by the Voyage 200. The same calculations done in figure 3b with “Complex format” mode set to rectangular instead of real. This was a dramatic example resulting in a false answer. In figure 3c we see that Nspire gives the correct answer in less than 1 second compared with the false answer given by the Voyage 200 after almost 60 seconds of computation.

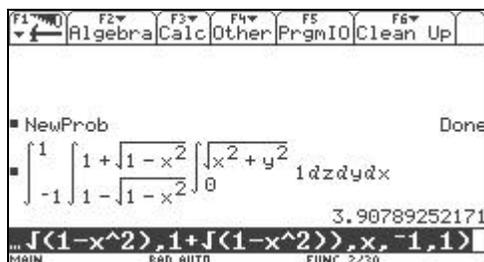


Figure 3a

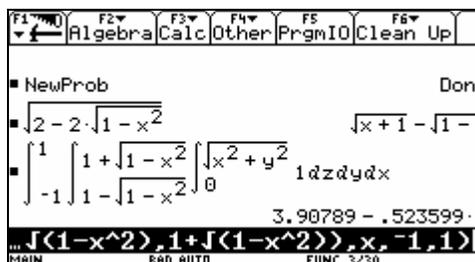


Figure 3b

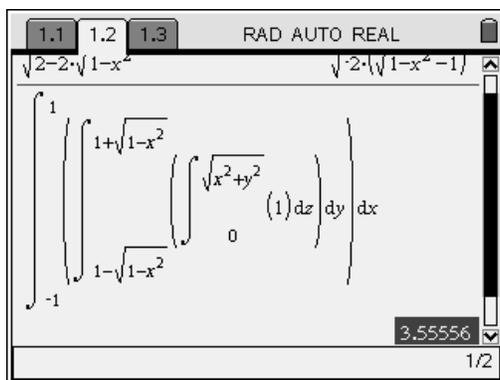


Figure 3c

As for example 3.2, it would be nice for Nspire to do like other CAS software and check for the form $e^{f(x)} \cdot f'(x)$ in the integrand (or for the more general case $f(g(x)) \cdot g'(x)$ for basic f functions), leading to the obvious solution by substitution as we teach our students to do. The fact that this CAS doesn't find a solution to this problem is surprising, given that the general idea of the substitution technique is implemented. The calculator has no problem if we replace the exponential function with a sine function, see figure 4, but will fail again if we use the square root function.

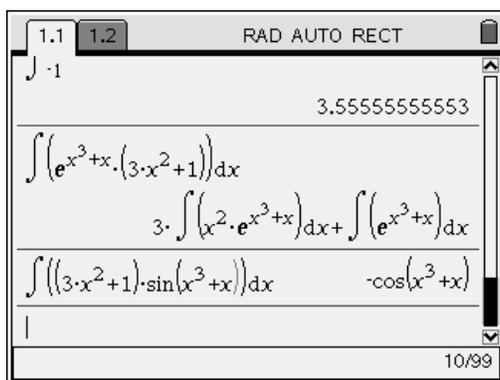


Figure 4

4. Problems with the solving facilities

Unfortunately, all but one problems in table 3 are still not fixed in Nspire, both handheld and software. Among all of these, the most dramatic one is in example 4.4: the answers to a system of equations should not depend on the choice of the variables. Let's recall, in tables 3a and 3b, these problems.

The problem with Voyage 200 OS 3.10	Is the problem fixed with Nspire CAS handheld OS 1.4.11643?	Is the problem fixed with Nspire CAS computer software OS 1.4.11654?
Example 4.1: (not really a bug). $\cos^{-1}\left(\frac{\sqrt{5}}{5}\right) + \tan^{-1}\left(\frac{1}{2}\right)$ is not simplified into $\frac{\pi}{2}$.	No, not simplified.	No, not simplified.
Example 4.2: $\sin(x^2) = 0$ is correctly solved but if we restrict the domain of x to -3 to 3 , we get 3 answers still involving $@n1$. We should get 5 answers: $0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}$.	No, the device still isn't able to give a clean answer to this equation for the restricted domain.	No, the device still isn't able to give a clean answer to this equation for the restricted domain.
Example 4.3: solving $\text{sys}t1: \begin{cases} \sin y = 0 \\ x \cos y = 0 \end{cases}$ The answer contains $x=0, y=@n1 \cdot \pi$ but also a second impossible solution. Solving $\text{sys}t2: \begin{cases} a \cdot n \cdot 2^{n-1} = 3 \\ a \cdot n \cdot 4^{n-1} = 24 \end{cases}$ only gives a floating point answer, except if we solve, first, for a .	No (but the word "when" is replaced by "{"). No.	No (but the word "when" is replaced by "{"). No.

Table 3a

In the case of example 4.2, we did notice some differences comparing answers from the Voyage 200 and from Nspire. With the Voyage 200, on the restricted domain, we get only 3 of the 5 answers (missing the 2 negative answers) and get a restriction on an arbitrary integer constant which obviously comes from the general solution and no longer is needed. (See figure 5a) With Nspire, we still get the comment on the arbitrary constant but at least, we get the 5 pertinent answers. (See figure 5b).

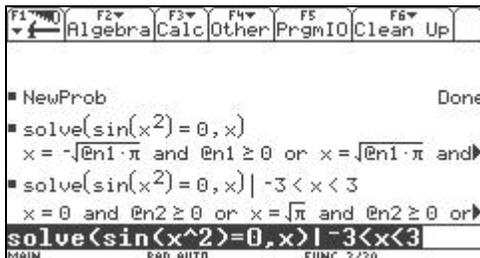


Figure 5a

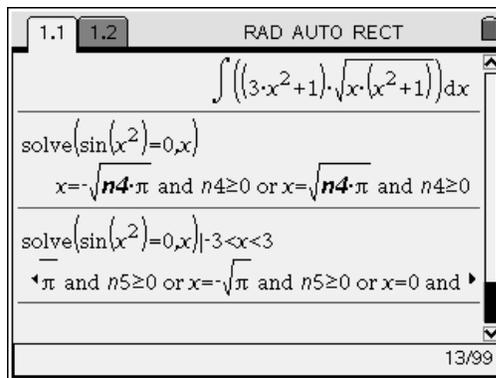


Figure 5b

<p>Example 4.4: 2 examples</p> <p>With $w1 = 100$, $w2 = 80$, $\alpha = \theta = 45^\circ$, consider the following non linear system of equations where tab, tac, tbd and β are the variables.</p> $andy11: \begin{cases} -tac \cos \alpha + tab \cos \beta = 0 \\ tac \sin \alpha + tabs \sin \beta - w1 = 0 \\ -tab \cos \beta + tbd \cos \theta = 0 \\ -tab \sin \beta + tbd \sin \theta - w2 = 0 \end{cases}$ <p>Zeros(<i>andy11</i>, {<i>tab</i>, <i>tac</i>, <i>tbd</i>, β}) returns the empty list but changing β for y gives an answer! But the solve command gives the answer no matter we use β or y.</p> <p>-----</p> <p>With ma, mb, g, and us being 4 positive parameters, consider the following non linear systems</p> $andy12: \begin{cases} -T + us * Na + ma * g * \sin \theta = 0 \\ Na - ma * g * \cos \theta = 0 \\ -T - us * Na + mb * g * \sin \theta = 0 \\ Nb - Na - mb * g * \cos \theta = 0 \end{cases}$ <p>We want to solve with respect to θ, T, Na and Nb. The zeros function gives the empty set but changing θ for a gives an answer.</p>	<p>No. The situation is exactly like before!</p> <p>see figure 6a, 6b</p> <p>No. The situation is exactly like before!</p> <p>see figure 6c</p> <p>No. The situation is exactly like before!</p> <p>No. The situation is exactly like before!</p>	<p>No. The situation is exactly like before!</p> <p>see figure 6c</p> <p>No. The situation is exactly like before!</p>
<p>Example 4.5:</p> $\text{solve} \left(m(s^2 X - s x(0) - x'(0)) + b(sX - x(0)) + kX = \frac{6s}{s^2 + 9} \right)$ <p>$\rightarrow chantal(m, b, k, x0, v0)$</p> <p>This function was not working properly for some values of the parameters (x was not found at the first time for some values).</p>	<p>Yes!</p>	<p>Yes!</p>

Table 3b

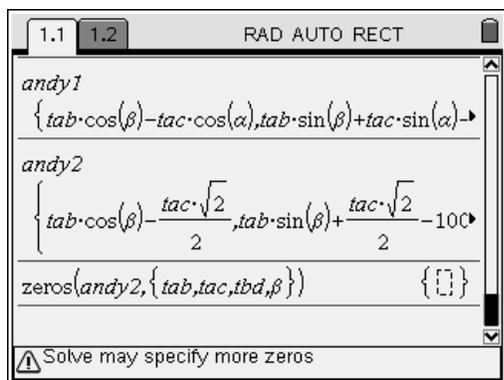


Figure 6a

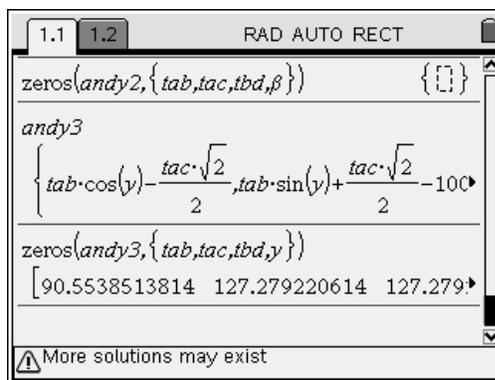


Figure 6b

The Nspire handheld has the same problem the Voyage 200 had. The solving of the system with the zeros() command fails, producing an empty list. If we substitute the variable y for β , we get the correct answer. (See figure 6a and 6b) Using the solve() command instead leads to the right answer in both cases. This is what we had observed with the Voyage 200 in our 2006 paper but we didn't get, with Nspire, the odd (but equivalent) following answer using this form of the command: solve({equation1,equation2,...},{list of variables}):

$$tab = 90.5539 \text{ and } tac = -128.062(\sin\beta - 1.10432) \text{ and } tbd = 127.279 \text{ and } \beta = 0.110657$$

Figure 6c below shows the solutions of both systems with the software version of Nspire. We see that in the andy12 system of equations, substituting variable a for θ will lead to an answer instead of the empty set.

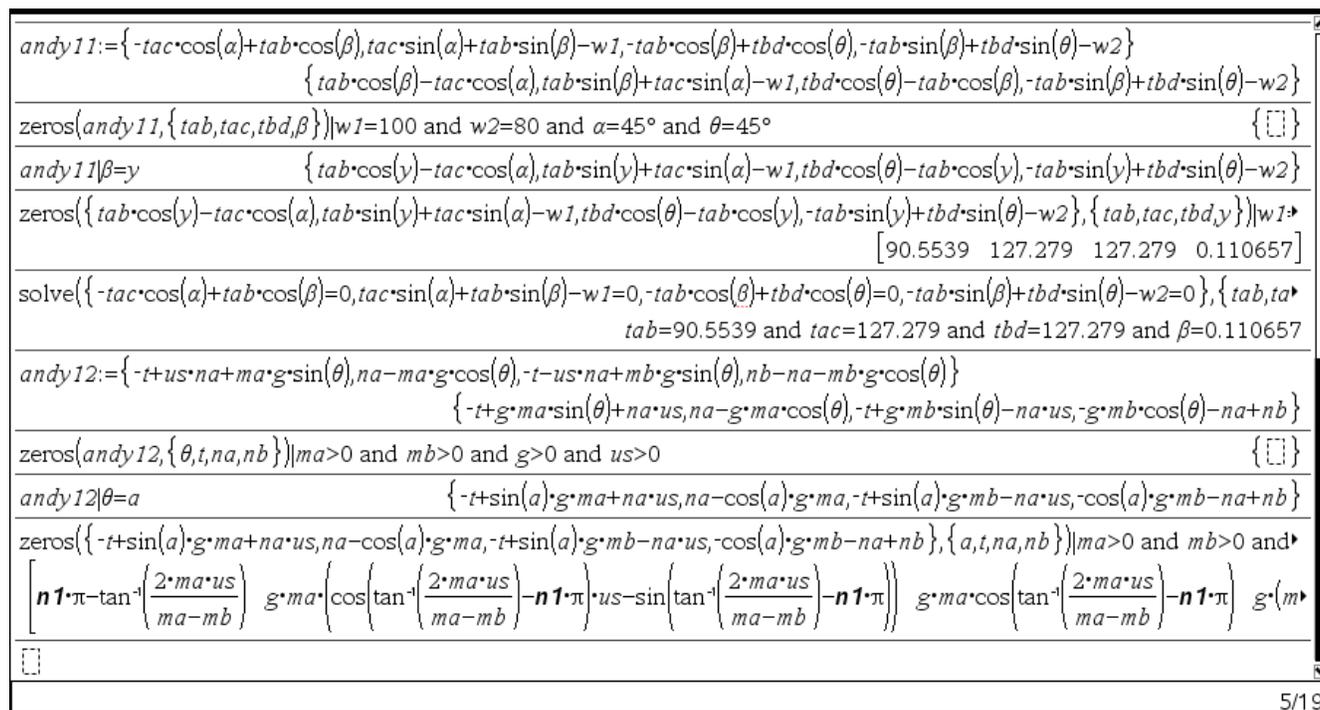


Figure 6c

This is a major problem; solutions of a system of equations should not depend on a choice of variable names in these equations. If we get an empty set showing no solutions for a given system of equations, should we try changing variable names just to be sure there is a really no solutions? This issue must be resolved!

5. New Problems (last 2 years)

Example 5.1: When computing, with the Voyage 200, the indefinite integral $\int \tanh(7x) dx$, we decided to abort because it was too long. The same thing happens with the Nspire handheld and with the computer version. With the calculators, we press the “On” key to break (or abort) calculations that are too long. With the software version, we finally learned to do the same thing holding down for a few seconds the “Pause/Break” key of the keyboard (or F12 on our notebook keyboard!).

With the Voyage 200, expanding the integrand first then gives the correct answer : $\frac{\ln(e^{14x} + 1)}{7} - x$ (see figure 7a). Using the “PropFrac()” command instead of “Expand()” gives a quicker answer with a more natural writing of the exponential function. But what is really surprising is the fact that the calculator easily gets the answer if we replace the constant 7 by a parameter a . As if the Voyage 200 is capable of doing a more general computation. (See figure 7b) The same thing happens with Nspire.

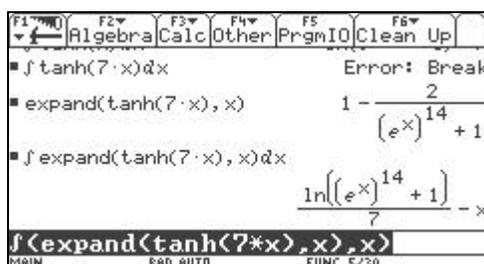


Figure 7a

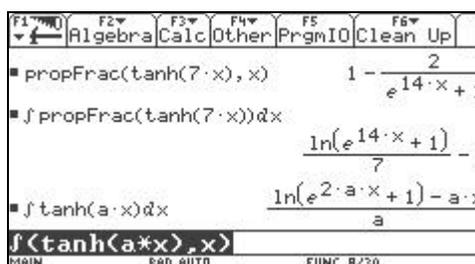


Figure 7b

Hyperbolic functions are more commonly used today; many of our engineering students will have to work with these functions. The Voyage 200 and Nspire do recognize these functions but that’s about all they can do. Given the right combination of exponential functions, the calculator will transform the expression to one using hyperbolic functions. This is true for the 3 basic hyperbolic functions ($\sinh(x)$, $\cosh(x)$ and $\tanh(x)$); the other hyperbolic functions are always transformed back to the 3 basic ones. If you need to work with these functions, the CAS will transform them back to exponential forms for calculations. This seems to be the problem with the bug mentioned in this example. In the following 7c figure, you can see how Nspire expands $\tanh(3x)$; we then

integrate this result. You can see that the result is the same as the one obtained by directly integrating this function. It is interesting to note that if the CAS system would convert the hyperbolic function using the “Expand()” command using the optional variable designation “Expand(expr,var)” instead of “Expand(expr)”, we would get easily the correct answer in a more compact form like the general case seen in figure 7b.

expand(tanh(3*x))	$\frac{2 \cdot (e^x)^2}{3 \cdot ((e^x)^4 - (e^x)^2 + 1)} - \frac{4}{3 \cdot ((e^x)^4 - (e^x)^2 + 1)} - \frac{2}{3 \cdot ((e^x)^2 + 1)} + 1$
$\int \left(\frac{2 \cdot (e^x)^2}{3 \cdot ((e^x)^4 - (e^x)^2 + 1)} - \frac{4}{3 \cdot ((e^x)^4 - (e^x)^2 + 1)} - \frac{2}{3 \cdot ((e^x)^2 + 1)} + 1 \right) dx$	$\frac{\ln\left(\left(e^{2 \cdot x} + 1\right) \cdot \left(e^{2 \cdot x} + \sqrt{3} \cdot e^x + 1\right) \cdot \left e^{2 \cdot x} - \sqrt{3} \cdot e^x + 1\right \right)}{3} - x$
$\int (\tanh(3 \cdot x)) dx$	$\frac{\ln\left(\left(e^{2 \cdot x} + 1\right) \cdot \left(e^{2 \cdot x} + \sqrt{3} \cdot e^x + 1\right) \cdot \left e^{2 \cdot x} - \sqrt{3} \cdot e^x + 1\right \right)}{3} - x$
expand(tanh(3*x),x)	$1 - \frac{2}{(e^x)^6 + 1}$
$\int \left(1 - \frac{2}{(e^x)^6 + 1} \right) dx$	$\frac{\ln(e^{6 \cdot x} + 1)}{3} - x$
$\int (\tanh(5 \cdot x)) dx$	$\frac{\ln(e^{10 \cdot x} + 1)}{5} - x$

Figure 7c

Now if you ask Nspire to expand (without the optional variable argument) $\tanh(7x)$, the resulting monster exponential expression can explain the difficulty of performing the integration. Strangely however, Nspire was able to find the simplified answer for the integration of $\tanh(5x)$ (needed 15 seconds of calculations on the handheld version).

Being more commonly used nowadays, these functions could be added to basic tables for derivatives and integral calculations. This is what MAPLE does. If you want the derivative of $f(x) = x \sinh(x)$, you could then obtain with the product rule the answer in terms of hyperbolic functions, $\sinh(x) + x \cosh(x)$. Nspire, like DERIVE and the Voyage 200, will of course give an equivalent answer in terms of exponential functions.

Example 5.2: If you try to find the critical points of $\sin(x^2 + y^2)$, you get an “Argument mismatch” error message. But, if you give a starting point to the solver, you get an answer. In figure 8a and 8b, you can see that we have the same problem with the Voyage 200 and Nspire. This non-linear system to be resolved is not that

complicated and the calculators should at least find the trivial solution $\{x, y\} = \{0, 0\}$. We cannot understand the reason for the “Error: Argument mismatch” message.

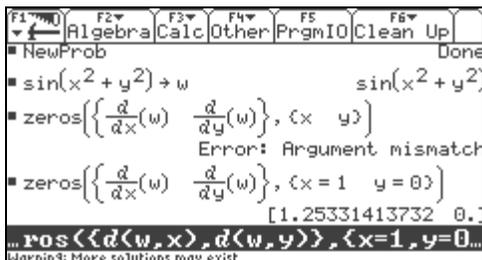


Figure 8a

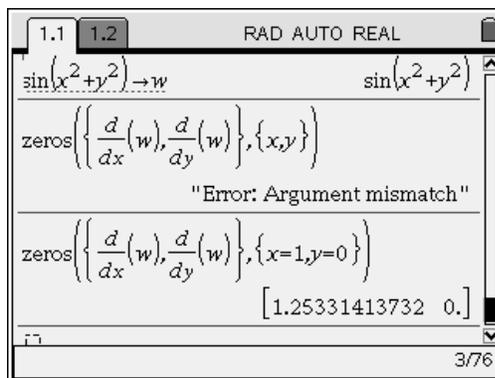


Figure 8b

Example 5.3: In AUTO mode, the fmin and fmax functions sometimes don't work correctly but do work correctly with \blacklozenge Enter. For example, consider the expression $x^4 - xy^2 + 3y^3 - 5y$.

Substitute for x the expression $1 + 2 \cos(t)$ and substitute for y the expression $1 + 2 \sin(t)$. The minimum of the new expression, in the interval $[0, 2\pi]$ is not achieved at endpoints as a graph shows (see figure 9b), but near $t = 3.35125$. We see in figure 9a that the fmin function wrongly gives the two endpoints for the minimum of this expression while with \blacklozenge ENTER, the Voyage 200 finds 3.351245, which is correct. The same problem with the fmax function will generate a good answer in AUTO mode, getting the two endpoints of the interval for the maximum value. But with \blacklozenge ENTER the calculator finds 1.667276 which is a value for a local maximum, not a global maximum. (See figure 9c)

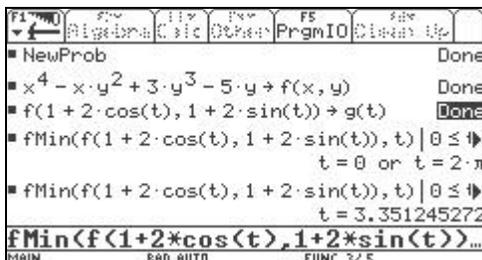


Figure 9a

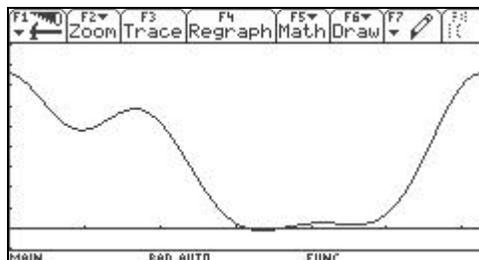


Figure 9b

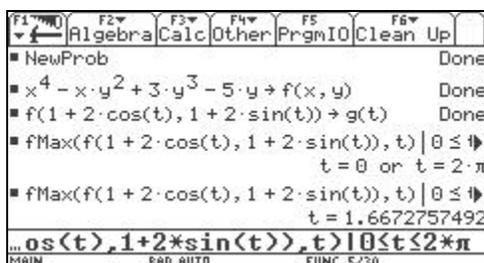


Figure 9c

Nspire, handheld and computer version, will do exactly the same as the Voyage 200, wrongly giving minimum of the function at the two endpoints, in AUTO mode, and giving a local maximum instead of the solution for a global maximum. However, there are two new functions available in the “Calculus...Numerical Calculations” menu, essentially numerical fmin and fmax functions, which will give the same answers as doing \blacklozenge ENTER on the classical fmin and fmax functions. (See figure 9d)

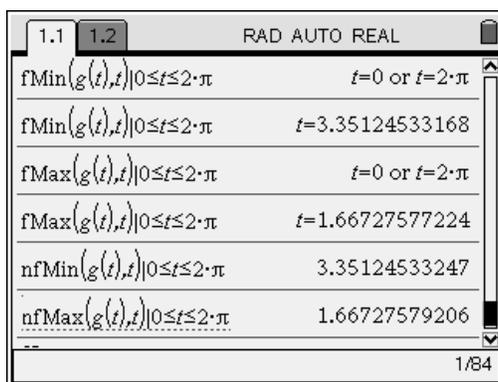


Figure 9c

Example 5.4: In this next case, we see a problem where the CAS system isn’t able to perform a simple indefinite integral. The Voyage 200 as well as Nspire can’t compute $\int (x+1)^{p+1} dx$. Every Calculus students

should easily find the antiderivative $\frac{(x+1)^{p+2}}{p+2}$, given of course that $p \neq -2$. These CAS systems seem to

decompose the integrand, $(x+1)^{p+1} = x(x+1)^p + (x+1)^p$, thus given the result shown in figure 10a.

Strangely, if you take $p-1$ as the exponent, the calculator has no problem finding the answer. Even substituting π for the constant p changes nothing in this situation, except with \blacklozenge Enter which will give the correct answer. In figures 10b and 10c, you can see strange behaviour for this problem with the exponent being $p+k$.

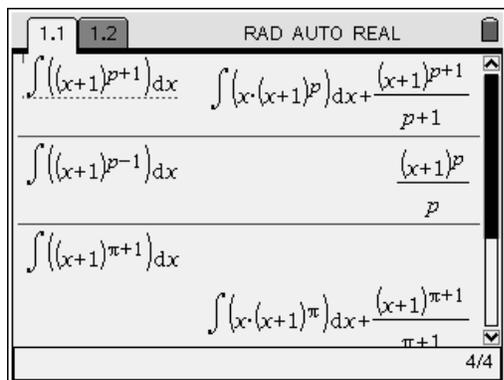


Figure 10a

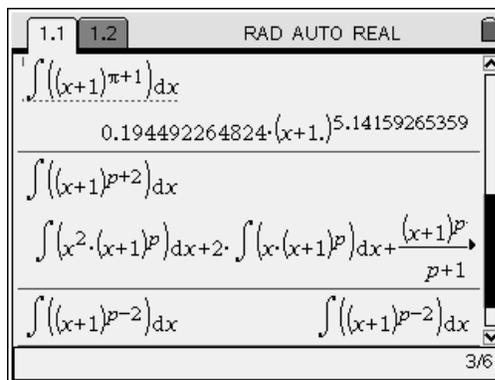


Figure 10b

Solving the integral of $(x+1)^{p+k}$ will duplicate the problem first mentioned if k is any positive integer. If k is any negative integer other than -1 , the CAS does nothing and you will get a correct answer if k takes any other real value.

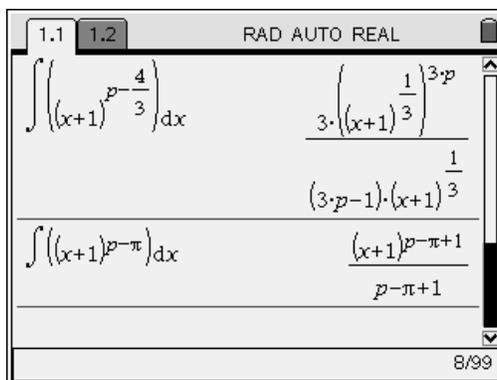


Figure 10c

These CAS systems should be able to do this basic problem, either by checking a simple substitution of variable to get back to the x^n formula of integration or even having a more complete table containing the antiderivative of $(ax+b)^n$.

Example 5.5: An engineering student at our university came to see us with a definite integral (from an avionics course) his Voyage 200 couldn't compute: $\int_0^c 4 \sin^2 \left(2 \cos^{-1} \left(1 - \frac{x}{c} \right) \right) dx$. The fact that his Voyage 200 isn't able to perform the integration is not necessarily a problem, this is not an easy integral to begin with.

We were curious of how other CAS system would react to this problem. As you can see in figure 11a, MAPLE (version 9.5 and version 12) isn't able either to find the solution.

```
> int((x+1)^(p+1),x);
```

$$\frac{(x+1)^{(p+2)}}{p+2}$$

```
> int(4*sin(2*arccos(1-x/c))^2,x=0..c);
```

$$\int_0^c 4 \sin\left(2 \arccos\left(-1 + \frac{x}{c}\right)\right)^2 dx$$

Figure 11a

DERIVE on the other hand had no problem getting the right answer $\frac{32c}{5}$.

```
#1:  $\int_0^c 4 \cdot \text{SIN}\left[2 \cdot \text{ACOS}\left(1 - \frac{x}{c}\right)\right]^2 dx$ 
```

```
#2:  $\frac{32 \cdot c}{15}$ 
```

Figure 11b

It is interesting to note that with DERIVE's "Display step" command in the "Simplify" menu, this problem requires a large number of steps (over 20) to get to the final answer. Since the Voyage 200 and the Nspire systems give the same results, we illustrate in figure 11c how we solved the problem with the student. We can see on the last line of the figure that Nspire would give a good answer if we "texpand()" the integrand. Why can't the CAS system try to do this operation (or "Expand()" or "PropFrac()") or ... depending on the problem) before given back the original problem with no solution? This is a good example showing why we still need to teach all the basic mathematics tools to our students.

$\int_0^c \left(4 \left(\sin \left(2 \cos^{-1} \left(1 - \frac{x}{c} \right) \right) \right)^2 \right) dx$	$4 \cdot \int_0^c \left(\left(\sin \left(2 \sin^{-1} \left(\frac{x-c}{c} \right) \right) \right)^2 \right) dx$
$4 \cdot (\sin(2 \cdot \theta))^2$	$4 \cdot (\sin(2 \cdot \theta))^2$
$\text{tExpand}(4 \cdot (\sin(2 \cdot \theta))^2)$	$16 \cdot (\sin(\theta))^2 \cdot (\cos(\theta))^2$
$16 \cdot (\sin(\theta))^2 \cdot (\cos(\theta))^2 _{\theta = \cos^{-1} \left(1 - \frac{x}{c} \right)}$	$\frac{-16 \cdot x \cdot (x - 2 \cdot c) \cdot (x - c)^2}{c^4}$
$\int_0^c \left(\frac{-16 \cdot x \cdot (x - 2 \cdot c) \cdot (x - c)^2}{c^4} \right) dx$	$\frac{32 \cdot c}{15}$
$\int_0^c \left(\text{tExpand} \left(4 \left(\sin \left(2 \cos^{-1} \left(1 - \frac{x}{c} \right) \right) \right)^2 \right) \right) dx$	$\frac{32 \cdot c}{15}$

Figure 11c

6. Conclusion

The majority of the issues we had presented in 2006, concerning the CAS portion of the Voyage 200, are still present in the new Nspire products. These are real problems encountered while using the calculator with our students. There were some problems fixed but clearly, in the last two years, the CAS portion of the new Nspire has not been the main focus of development for this product. In the majority of the cases we studied, the new Nspire calculators and software seem only to duplicate (at a much higher speed, we do admit) the working of the Voyage 200. Fixing some of the problems presented in this paper should not be a difficult task and would denote a real commitment to developing the CAS portion of this product. We would also appreciate more focus on the needs for a university level usage in mathematics and science courses. The Voyage 200 actually does a great job for us, just fixing some of the bugs encountered, adding some new functions like we saw in the Nspire and getting a faster processor would be a more appealing perspective for our teaching of math and sciences.